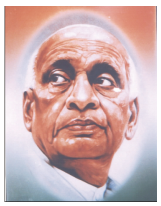
POINT SET TOPOLOGY

An Experience of a Teacher

Dinesh Karia



Dedicated to
the Integrator
of India

Bakrol-Anand

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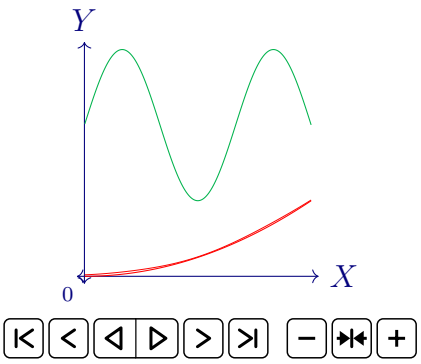

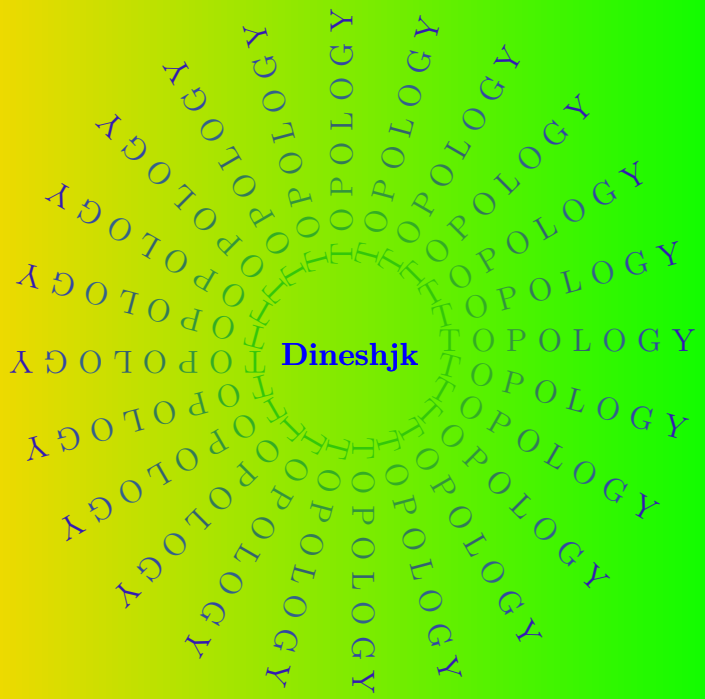

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Preface

The handwritten notes on Point Set Topology started in 1990 in two parts covering first and the second course at M.Sc. It was in 2011 that I started preparing this notes using LaTeX for the First Course on Topology at masters under the *Choice Based Credit System*. In October 2018, after retirement, I left the boundaries of the prescribed syllabus by adding material to the notes. My idea was to evolve a note on Point Set Topology, over a period of time, that can be used as self-reading notes. I hope the reader will find these notes suitable for self-reading. Just as the homosapiens is not the final stage of evolution, no version of these notes will be final. But at some stage, adding new sections will have to come to rest. Instead, we may go for the next book on Algebraic Topology. All in all, I honestly believe, this is a sincere effort to create a learning aid.

I used to enthuse the students to ask questions and students did ask. However, in the very first year of my teaching, Mr. Anil J. Patel, a student of second course, asked, "If a space is Hausdorff, then all its compact subsets are closed. Does the converse hold?" That is, in other words, is it necessary for a space to be Hausdorff if all its compact subsets are closed? I realized at that time that not only a physical labourer but a postgraduate teacher also gets sweating as a part of his/her work. It took about two weeks for me to answer the question. The counter example I found after a lot of struggle was the cocountable topology on an uncountable set. The topology became my favourite and later on I could also replace the example of $\mathbb{R}^{\mathbb{R}}$ of Willard [**Wil70**] by the cocountable topology to demonstrate the inadequacy of sequences to characterize closures as well as continuity. My respect towards the cocountable topology increased a lot.

Teaching and authoring are, without any doubt, the best ways to learn. New ideas pop up during both processes. While typesetting this book, I could realize that parallel to the cofinite and the cocountable topologies, we can have two topologies associated to each infinite cardinal. This class of the *cardinal topologies* are presented in Section 7 and based on the same model, the α-nets sitting between the sequences and classical nets are discussed in Section 34. **The concept has not appeared in the literature so far, to the best of my knowledge.**

The notes contain many exercises. In fact, when time permits and the mood swings, I type solutions to the exercises too. However, I am a life long teacher-cum-learner and wish the reader to solve the exercises themselves. The solutions are kept hidden from the processed output file. If you are stuck at any point while solving a particular exercise, you are most welcome to communicate your queries to me at the email address hidden behind the word **FreeBird** at the end of this preface. I will make all efforts to reply.

The knowledge is not a copyright of anybody, and I am that anybody. Anyone can use and include the excerpts from this note. **However, a due acknowledgement will be appreciated, though it is not obligatory.**

At the outset, I should acknowledge that I adopted new style of Chapter and colourful boxed theorem style from **Dr. Jay Mehta**, my student turned colleague, while typesetting my class notes on Group Theory as well as updating class notes on Topology in 2017.

Later on, as an experiment, I invited a few students turned colleagues to add to my notes. I am happy to note that Jay gave a couple of points in TEX that I included. I gratefully acknowledge the same here. One can find the credits, in two footnotes to two items he contributed.

Any typo in these notes does not count towards the wrong mathematics. It is the responsibility of the readers to correct it and understand. In fact, if one understands the mathematics, the typo, if any, will be revealed. Let me invite all the readers of this notes to bring to my notice any typo or flaw in the text.

Dinesh J. Karia
FreeBird
Notes started on: 31 October 2018
Sardar Jayanti

Take Off

A Brief History

Topology intrudes in almost all areas of mathematics – ranging from Functional Analysis to Relativity and Cosmology. The topology of various networks of computer is nothing but an application of Topology. Topology is further divided into various branches like Point Set topology, algebraic topology and differential topology. As the abstract study of Topology began in early 20th century, mathematicians identified the branch as a new language of mathematics. Later on Functional Analysis, a marriage of Topology with Linear Algebra, took over to be the Queen and the language of mathematics. Euler was the first to use topology giving the name *geometry of position*. He solved the open problem of **Königsberg bridges** in one of his research papers. Euler then followed to give his famous formula $V - E + F = 2$ for convex polyhedra. Later, the formula was improved upon by L'Huilierin 1813 as $V - E + F = 2 - 2g$ encompassing all orientable polyhedra. Here g is called the *genus*, the integer representing the maximum number of cutting along a closed simple path leaving the original surface connected. (For example, if we have a sphere, then cutting the sphere along any closed path on it makes the sphere into two pieces. So the genus of a sphere is 0). $V - E + F = 2 - 2g$ is the first formula describing the topological invariant. At times, the treasure of topological results was enriched by Bolzano (1781–1848), Möbius (1790–1868), Weierstrass (1815–1897), Betti (1823–1892), Riemann (1826–1866), Jordan (1838–1922), Cantor (1845–1918), Poincaré (1854–1912), Hilbert (1862–1943), Hausdorff (1868–1942), Heegaard (1871–1948), Fréchet (1878–1973) and scores of other mathematicians. Topology is chiefly divided into four subfields (1) General Topology or Point Set Topology (2) Algebraic Topology (3) Differential Topology and (4) Geometric Topology. The present book deals with very primary Point Set Topology.

With a rapid development of Topology in twentieth century, a vast variation has been seen in the terminology adopted by various authors. There are a number of terms swapped with each other adding to the burden to the new entrants in the course. For example some authors consider a *neighbourhood* to be open by definition, whereas some don't. Just like that some authors define T_3-spaces to be combination of *regularity* and T_1-ness. T_3 of these mathematicians becomes regular for others and their *regular spaces* become T_3-spaces for the rest. The same story is shared by T_4 and *normal spaces*. Though keeping the consistency throughout a course of a book does not harm the development of the subject. There are two definitions of *locally compact spaces*, both of which agree if a space is Hausdorff.

However, according to one, a compact space turns out to be locally compact for the other it does not.

Serious Trivia

- ☞ The words *tópos* ($\tau\acute{o}\pi o\zeta$) and *logos* ($\lambda o\gamma o\zeta$) are Greek words meaning a *place* and *discourse* respectively.
- ☞ Listing was the first mathematician to use the word *Topologie* in German in 1847 in his formal publication. In his informal communications and notes he had been using the word for more than 10 years then.
- ☞ When Topology was first properly founded toward the end of the 19th century, it was called *geometria situs*, a Latin phrase meaning *geometry of place* and also *analysis situs*, a Latin word meaning *analysis of place*. From around 1925 to 1975 it was an important growth area within mathematics.
- ☞ The word *Topology* was first used in English by a Scottish mathematician Tait in his article [**Tai83**].
- ☞ The very first abstract definition of topology was given by Hausdorff in which the T_2-condition was a part of the definition.
- ☞ Hilbert gave the idea of *neighbourhood* in 1902. However the word *neighbourhood* was first used by Cathcart in the English translation of a German book [**VH81**] by von Harnack.
- ☞ Fréchet introduced the concept of *metric spaces* in his Doctoral Thesis in April 1906. However the phrase *metric space* for these objects, was first used by Felix Hausdorff.
- ☞ The concepts of a real-valued *semicontinuous function* defined on $\mathbb{R}$ and *sets of first and second categories*, were first introduced by Baire [**Bai99**]. However, before this he had described his research on all this in the letters written by him to Borel.
- ☞ Fréchet also defined *compact spaces* first with the help of limit points. Today's definition of compact space was introduced by Alexandroff and Urysohn.
- ☞ Sorgenfrey provided a topology on $\mathbb{R}$ with the name of *lower limit topology* that was christened after him later on. $\mathbb{R}$ with the lower limit topology is called the *Sorgenfrey line* and $\mathbb{R}^2$ with the product of the lower limit topology is called the *Sorgenfrey plane*. It was the first counter example showing that product of normal spaces need not be normal. Later on it served as a typical or counter example in topology in number of aspects.
- ☞ Separation conditions like T_0, T_1, T_3, T_4 etc. were given by Tychonoff.
- ☞ Zermeloproved the Theorem 6.27 in 1904 only to attract criticism on his *non-constructive proof*. He later on proved it again in 1908 in his paper *Neuer Beweis* meaning New Evidence. Of course his theorem is equivalent to the Axiom of Choice (Theorem 4.5).
- ☞ Hausdorff, on being informed to have been interned, committed suicide on 26 January 1942 together with his wife and wife's sister [**Mac20**, Hausdorff]. He wrote the following suicidal note addressed to his friend.

  Dear Friend Wollstein,
  By the time you receive these lines, we three will have solved the problem in another way – in the way which you have continually attempted to dissuade us. . . .

What has been done against the Jews in recent months arouses well-founded anxiety that we will no longer be allowed to experience a bearable situation. . . .
Forgive us, that we still cause you trouble beyond death; I am convinced that you will do what you are able to do (and which perhaps is not very much). Forgive us also our desertion! We wish you and all our friends will experience better times
Yours faithfully
Felix Hausdorff

☞ There are eight classes of homeomorphic English Alphabet written in uppercase sans serif font, *viz.*
$\{\mathsf{A},\mathsf{R}\}$, $\{\mathsf{B}\}$, $\{\mathsf{C},\mathsf{G},\mathsf{I},\mathsf{J},\mathsf{L},\mathsf{M},\mathsf{N},\mathsf{S},\mathsf{U},\mathsf{V},\mathsf{W},\mathsf{Z}\}$, $\{\mathsf{D},\mathsf{O}\}$, $\{\mathsf{E},\mathsf{F},\mathsf{T},\mathsf{Y}\}$, $\{\mathsf{H},\mathsf{K}\}$, $\{\mathsf{P},\mathsf{Q}\}$ and $\{\mathsf{X}\}$.

Standard Sets

No.	Symbol	Description of the set
(i)	$\mathbb{N}$	The set of all natural numbers
(ii)	$\mathbb{Z}$	The set of all integers
(iii)	J_n	$\{1,2,3,\ldots,n\}$, $n\in\mathbb{N}$
(iv)	$\mathbb{Q}$	The set of all rational numbers
(v)	$\mathbb{R}$	The set of all real numbers
(vi)	$\mathbb{C}$	The set of all complex numbers
(vii)	$\mathbb{R}\times\mathbb{R}$	$\{(x,y):x,y\in\mathbb{R}\}$
(viii)	$\mathbb{R}^n$	$\{(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\ldots,\mathbf{x}_n):\mathbf{x}_i\in\mathbb{R},1\le i\le n\}$, $n\in\mathbb{N}$
(ix)	(a,b)	$\{x\in\mathbb{R}:a<x<b\}$, $-\infty\le a<b\le\infty$
(x)	$[a,b]$	$\{x\in\mathbb{R}:a\le x\le b\}$, $-\infty<a<b<\infty$
(xi)	$(a,b]$	$\{x\in\mathbb{R}:a<x\le b\}$, $-\infty\le a<b<\infty$
(xii)	$[a,b)$	$\{x\in\mathbb{R}:a\le x<b\}$, $-\infty<a<b\le\infty$
(xiii)	S^1	$\{(x,y)\in\mathbb{R}^2:x^2+y^2=1\}$
(xiv)	S^n	$\{(\mathbf{x}_1,\ldots,\mathbf{x}_{n+1})\in\mathbb{R}^{n+1}:\sum_{i=1}^{n+1}\mathbf{x}_i^2=1\}$

TABLE 1. Standard Sets

CHAPTER 1

Basic Set Theory.

Whenever we talk of sets, we fix a set, called the *uninversal set*, and all the elements are coming from that set. We assume the reader to be familiar with the set theory covered in a school course. This includes *empty set*, *subset*,[1] *superset*, *finite union* and *finite intersection* of sets, *complement*, *difference and symmetric difference of two sets*. We begin with the cartesian product and relations.

1. Cartesian Product and Relations

1.1. Definition. Let X, Y be sets. The set

$$X \times Y = \{(x, y) : x \in X, y \in Y\} \tag{1.1.1}$$

is called the *cartesian product of X and Y*. For $x \in X, y \in Y$, (x, y) is called an *ordered pair*.

The definition of the cartesian product can be extended to finite cartesian product.

1.2. Definition. Let $X_1, X_2, X_3, \ldots, X_n$ be sets. The set

$$\prod_{i=1}^{n} X_i = \{(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n) : \mathbf{x}_i \in X_i, 1 \leq i \leq n\} \tag{1.2.1}$$

is called the *cartesian product of $X_1, X_2, X_3, \ldots, X_n$*. It is also denoted by $X_1 \times X_2 \times X_3 \times \cdots \times X_n$. An element $(x_1, x_2, x_3, \ldots, x_n) \in \prod_{i=1}^{n} X_i$ is called an *n-tuple*

I. N$^o{}_t$ations: ☞ $X^n = \underbrace{X \times X \times X \times \cdots \times X}_{n\text{-times}}$.

The concepts of relation and function are central to the set theory.

[1] Note that $A \subset B$ also includes the possibility of equality.

1.3. Definition. For sets X, Y, a subset R of $X \times Y$ is called a *relation from X to Y*. If $X = Y$, then R is called a *relation on X*. For $(x, y) \in R$, we write xRy.

1.4. Definition. Let R be a relation from a set X to a set Y. Associated to R, there is a unique relation R^{-1} from Y to X given by

$$R^{-1} = \{(y, x) \in Y \times X : xRy\}.$$

R^{-1}, read as R inverse, is called the *inverse of R*.

It is obvious that $(R^{-1})^{-1} = R$.

1.5. Definition. Let X, Y be sets, $A \subset X, B \subset Y$, $a \in X$, $b \in Y$ and R be a relation from X to Y. The set

$$R(a) = R[a] = \{y \in Y : aRy\} \tag{1.5.1}$$

is called the *image of a* under R and the set

$$R(A) = R[A] = \{y \in Y : xRy \text{ for some } x \in A\} \tag{1.5.2}$$

is called the *image of A* under R. The set $R^{-1}(b) = R^{-1}[b]$ is called the *inverse image* or *preimage of b* under R. The set $R^{-1}(B) = R^{-1}[B]$ is called the *preimage of B* or *inverse image of B* under R.

1.6. Remark. Note that $R(a)$ in (1.5.1) and $R(A)$ in (1.5.2) are sets because any point $a \in X$ may be related to one, none or more than one element of Y.

1.7. Definition. Let X, Y, Z be sets, R be a relation from X to Y and S be a relation from Y to Z. The relation

$$S \circ R = \{(x, y) : \text{ there is } z \in X \text{ such that } xRz, zSy\}$$

is called the *composition* of R and S. In this case, we shall also write $(x, y) = (z, y) \circ (x, z)$ or $xRzSy$.

1.8. Definition. The relation

$$\Delta = \{(x, x) : x \in X\}$$

on a set X is called the *diagonal*. When the explicit context of X will be needed, we shall denote it by $\Delta(X)$.

Problems

1. Show that $\emptyset$ is a subset of every set.

2. If $A \subset B \subset C$, then show that $A \subset C$.

3. If $A \subset B$ and $C \subset D$, then prove the following.

(1) $A \cap C \subset B \cap D$. (2) $A \cup C \subset B \cup D$. (3) $A \times C \subset B \times D$.

4. For nonempty sets A, B, C, Discuss the validity of

(1) $A \cap B \subset A \cap C \Rightarrow B \subset C$.
(2) $A \cup B \subset A \cup C \Rightarrow B \subset C$.
(3) $A \times B \subset A \times C \Rightarrow B \subset C$.

5. Give an example of a relation R from a set X to a set Y and two distinct subsets $A_1, A_2 \subset X$ and two distinct subsets $B_1, B_2 \subset Y$ such that $R(A_1) = R(A_2)$ and $R^{-1}(B_1) = R^{-1}(B_2)$.

2. Functions

The concept of function is omnipresent in mathematics and its applications everywhere in basic sciences, social science and arts. The first encounter with the concept of a function is usually through the following rudimentary definition.

2.1. Definition. Let X, Y be two nonempty sets. A *function* f from X to Y is a rule assigning to each element x of X a unique element y of Y, written $y = f(x)$. Symbolically, a function f from X to Y is written as $f\colon X \to Y$. Also, X is called the *domain of* f and Y is called *the codomain of* f.

The next, in turn, is the following refined and traditional definition of a function. It introduces the same object as in the Definition 2.1 in a different way.

2.2. Definition. Let X, Y be nonempty sets. A relation f from X to Y is said to be a *function* if for every $x \in X$, there is a unique $y \in Y$ such that xfy. In this case, we write $y = f(x)$. Also, y is called the *image of* x *under* f or *value of* f *at* x.

The irony with this definition is that students perceive one of the two extremities: one common y for all x or distinct y for distinct x. There could be distinct images for distinct points for some elements of the domain and some elements may share a common image. We rephrase the Definition 2.2. It may make this point clear.

2.3. Definition. Let X, Y be two nonempty sets a relation f from X to Y is called a *function* from X to Y if for every $x \in X$, $f(x)$ is a singleton. A function is also called a *map* or a *mapping*.

Note

For a function $f\colon X \to Y$, writing $y = f(x)$ when xfy, is perfectly legitimate because for a given $x \in X$, we have one and only one $y \in Y$ such that xfy. However for a relation $R \subset X \times Y$, given $x \in X$, there may be none, one or more than one element of Y related to x via R, say for example there are $y, z \in Y$ such that xRy as well as xRz. In this case, writing $y = R(x)$ and $z = R(x)$ is a flaw.

2.4. Definition. Let X, Y, Z, W be sets and $f\colon X \to Y$ and $g\colon Z \to W$ be functions. We say that $f = g$ if

(1) $X = Z$,
(2) $Y = W$ and
(3) $f(x) = g(x)$ for all $x \in X$.

That is, if f and g have the same domains, same codomains and their values coincide at every point of the domain, then they are called the *equal functions*.

2.5. Definition. For a set X and $A \subset X$, the *characteristic function of A* is defined to be the function $\chi_A\colon X \to \mathbb{R}$,

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A; \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if $A = \{a\}$, a singleton, then χ_A is denoted by δ_a and is called the *point mass at a*.

2.6. Examples.

(1) $f\colon [0,1] \to \mathbb{R}$ and $g\colon [0,1] \to \mathbb{R}$ defined by $f(x) = x^2$ and $g(x) = x^3$ are different functions because they assign different values to some $x \in [0,1]$, say, for example, $f(0.5) = 0.25$ and $g(0.5) = 0.125$.
(2) $f\colon [0,1] \to \mathbb{R}$ and $g\colon [0,1] \to \mathbb{R}$ defined by $f(x) = x$ and $g(x) = |x|$ are the same functions because their domains are the same, their codomains are the same and they assign the same value x to every $x \in [0,1]$.
(3) $f\colon [-1,1] \to \mathbb{R}$ and $g\colon [-1,1] \to \mathbb{R}$ defined by $f(x) = x$ and $g(x) = |x|$ are different functions because they assign different values to some $x \in [-1,1]$, say, for example, $f(-0.5) = -0.5$ and $g(-0.5) = 0.5$.
(4) $f\colon [0,1] \to \mathbb{R}$ and $g\colon [-1,1] \to \mathbb{R}$ defined by $f(x) = x^2$ and $g(x) = x^2$ are different functions because, their domains are different.
(5) $f\colon [0,1] \to [0,1]$ and $g\colon [0,1] \to \mathbb{R}$ defined by $f(x) = x^2$ and $g(x) = x^2$ are different functions because, their codomains are different.

2.7. Definition. Let X, Y be sets and $f\colon X \to Y$ be a function. Then $f(X)$ is called *the range of f* or *image of f*.

2.8. Definition. Let X, Y be sets and $f\colon X \to Y$ be a function. The set

$$G(f) = \{(x, f(x) : x \in X)\} \subset X \times Y$$

is called the *graph of f*.

2.9. Definition. Let X, Y be nonempty sets and $f\colon X \to Y$ be a function. If $f(X) = Y$, then f is called an *onto function* or a *surjective function* or a *surjection*. If $x, y \in X, x \neq y \Rightarrow f(x) \neq f(y)$, then f is called a *one-one function* or a *one to one function* or an *injective function* or an *injection*. A function which is not one-one is called a *many to one* function. A one-one, onto function is also called a *bijective function* or a *bijection*.

2.10. Definition. Two nonempty sets X, Y are said to be in *one to one correspondence* if there is a bijective function $f\colon X \to Y$.

2.11. Definition. Let X be a set and $Y \subset X$. The function $\iota_Y\colon Y \to X$ defined by $\iota(x) = x$, $(x \in Y)$, is called an *inclusion map*. $\iota_X\colon X \to X$ is called the *identity map*. Usually ι_X is denoted by ι.

2.12. Definition. Let X, Y be two nonempty sets and $Z \subset X$. Let $f\colon X \to Y$ be a function. Define a new function $g\colon Z \to Y$ by $g(z) = f(z)$, $(z \in Z)$. Then g is called the *restriction of f (from X) to Z* and is denoted by $f_{|Z}$, and f is called an *extension of g (from Z) to X*.

2.13. Definition. For nonempty sets X, Y a function $f\colon X \to Y$ is said to be a *constant function* if there is $c \in Y$ such that $f(x) = c$ for all $x \in X$.

2.14. Remark. Let X, Y be nonempty sets and $f\colon X \to Y$ be a function.

(1) The identity map $\iota\colon X \to X$ is a one to one correspondence.
(2) In general, f^{-1} need not be a function. So, for $y \in Y$, $f^{-1}(y) \subset X$.
(3) If f is injective, then $f^{-1}\colon f(X) \to X$ is a function, called the *inverse of f*. In particular, when f is a bijection, $f^{-1}\colon Y \to X$ is a bijection.

2.15. Definition. Let X, Y, Z be nonempty sets, $f\colon X \to Y$ and $g\colon Y \to Z$ be two functions. Then the function $g \circ f\colon X \to Z$, called *composition* of f and g, is defined as $g \circ f(x) = g(f(x))$, $(x \in X)$.

2.16. Proposition. *Let X, Y be nonempty sets, $Z \subset X$ and $f\colon X \to Y$ be a function. Then $f_{|Z} = f \circ \iota_Z$.*

Proof. Clearly, Z is the domain of $f_{|Z}$ as well as $f \circ \iota_Z$ and Y is the codomain of both. Also for $x \in Z$, $f_{|Z}(x) = f(x) = f(\iota_Z(x)) = f \circ \iota_Z(x)$. □

2.17. Proposition. *Let X, Y, Z be nonempty sets, $f\colon X \to Y$ and $g\colon Y \to Z$ be functions. Then $g \circ f$ is one-one if and only if f and $g_{|f(X)}$ are one-one.*

PROOF. Suppose $g \circ f$ is one-one. For $a, b \in X$, $f(a) = f(b) \Rightarrow g(f(a)) = g(f(b)) \Rightarrow a = b$ because $g \circ f$ is one-one. Thus f is one-one. Also for $c, d \in f(X)$, suppose $g(c) = g(d)$. Choose $a, b \in X$ such that $f(a) = c$ and $f(b) = d$. Then $g(f(a)) = g(c) = g(d) = g(f(b)) \Rightarrow a = b \Rightarrow f(a) = f(b) \Rightarrow c = d$. Thus $g_{|f(X)}$ is one-one.

Conversely, suppose f and $g_{|f(X)}$ are one-one. For $a, b \in X$, $g \circ f(a) = g \circ f(b) \Rightarrow g(f(a)) = g(f(b))$. Since g is one-one on $f(X)$, $f(a) = f(b)$. But injectivity of f guarantees that $a = b$. This completes the proof. □

2.18. Proposition. *Let X, Y be nonempty sets $f\colon X \to Y$ be a function and $E, Z \subset X$. For $F \subset Y$,*

(1) $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$.
(2) $\iota_Z^{-1}(E) = Z \cap E$
(3) ${f_{|Z}}^{-1}(F) = Z \cap f^{-1}(F)$.

PROOF.

☞(1)
$$\begin{aligned} x \in f^{-1}(Y \setminus F) &\Leftrightarrow f(x) \in Y \setminus F \\ &\Leftrightarrow f(x) \notin F \\ &\Leftrightarrow x \notin f^{-1}(F) \\ &\Leftrightarrow x \in X \setminus f^{-1}(F) \end{aligned} \tag{2.18.1}$$

☞(2)
$$\begin{aligned} x \in \iota_Z^{-1}(E) &\Leftrightarrow \iota_Z(x) \in E \\ &\Leftrightarrow \iota_Z(x) \in E, x \in Z \\ &\Leftrightarrow x \in E, x \in Z \\ &\Leftrightarrow x \in Z \cap E \end{aligned} \tag{2.18.2}$$

☞(3) By (2), ${f_{|Z}}^{-1}(F) = (f \circ \iota_Z)^{-1}(F) = \iota_Z^{-1}(f^{-1}(F)) = Z \cap f^{-1}(F)$. □

Problems

6. In context of Exercise 5, give an example of a function f from a set X to a set Y and two distinct subsets $A_1, A_2 \subset X$ such that $f(A_1) = f(A_2)$. Can we have two distinct subsets $B_1, B_2 \subset f(X)$ such that $f^{-1}(B_1) = f^{-1}(B_2)$?

7. Give an example of a function f from a set X to a set Y and two distinct nonempty subsets $B_1, B_2 \subset Y$ such that $f^{-1}(B_1) = f^{-1}(B_2)$. Can we have an example having $f^{-1}(B_1) = f^{-1}(B_2)$ if $Y \setminus f(X)$ is at most singleton and $B_1 \cap B_2 = \emptyset$.

8. On a nonempty set X, can we have two distinct functions $f, g\colon X \to X$ satisfying any of the following conditions?

(1) $G(f) \cap G(g) \subset \Delta$.
(2) $\Delta \subset G(f) \cup G(g)$.
(3) $\Delta \subset G(f)$.

9. For nonempty sets X, Y, a function $f\colon X \to Y$, $A, B \subset X$ and $C, D \subset Y$. Prove the following.

(1) $f(A \cap B) \subset f(A) \cap f(B)$ and equality need not hold. If f is one-one, then show that equality holds.

(2) $f(A \cup B) = f(A) \cup f(B)$.
(3) $f^{-1}(C \cap D) = f^{-1}(C) \cap f(D)$.
(4) $f^{-1}(C \cup D) = f^{-1}(C) \cup f(D)$.
(5) $A \subset f^{-1}(f(A))$ and the equality need not hold. If f is one-one, then show that equality holds. Also, show that the equality holds for all $A \subset X$, then f is one-one.
(6) $f(f^{-1}(D)) \subset D$ and the equality need not hold. Show that the equality holds if and only if $D \subset f(X)$.

10. Let X, Y, Z be nonempty sets and $f\colon X \to Y$ and $g\colon Y \to Z$ be functions.

(1) Show that f, g are one-one $\Rightarrow g \circ f$ is one-one $\Rightarrow f$ is one-one. Also show that converse of any of these does not hold.
(2) Show that f, g are onto $\Rightarrow g \circ f$ is onto $\Rightarrow g$ is onto. Also show that converse of any of these does not hold.

3. Cardinality

In this section, we shall deal with the generalized numbers, called *cardinal numbers*. In what follows, we shall denote $J_n = \{1, 2, 3, \ldots, n\}$, where $n \in \mathbb{N}$. Cardinality or the cardinal number of a set is the measure of its size in terms of number of elements it contains.

3.1. Definition. Let X, Y be two sets. We say that X and Y have *the same cardinality*, and write $|X| = |Y|$, if there is a bijective function from X to Y. In this case, we also say that X and Y are *equipotent*.

Clearly, the following is apparent.

(1) Identity is a bijection from a set to itself. Thus $|X| = |X|$ for every set.
(2) Inverse of a bijection is a bijection. Thus if $|X| = |Y|$, then $|Y| = |X|$.
(3) Composition of two bijections is a bijection. Thus if $|X| = |Y|$ and $|Y| = |Z|$, then $|X| = |Z|$.

We say that the cardinality of X, is less or equal to the cardinality of Y if there is an injective function from X to Y. In this case, we write $|X| \leq |Y|$. We write $|X| < |Y|$ if $|X| \leq |Y|$ but $|X| \neq |Y|$.

3.2. Definition. We define $|\emptyset| = 0$, the first cardinal number. We also define $|J_n| = n$.

3.3. Definition. A set X is said to be a *finite set* if either $X = \emptyset$ or for some $n \in \mathbb{N}$, X is equipotent to J_n. In the latter case, we say that the *cardinality of X is n*. That is, $|X| = n$. If a set is not finite, then it is called an *infinite set*.

3.4. Definition. A set X is called *countably infinite* if X is equipotent to $\mathbb{N}$ and its cardinality is denoted by $\aleph_0$ (read *aleph not*). A set X is said to be *countable* or *at most countable* if X is either finite or countably infinite. A set which is not countable, is called an *uncountable set*.

3.5. Remark. From the preceding two definitions, one observes that the "cardinality" of a set is nothing but the number of elements in that set. Such numbers are called the *cardinal numbers*. Of course, $0, 1, 2, 3, \ldots$ are cardinal numbers. These are called *finite cardinal numbers*. However, there are many cardinal numbers other than these. They are all *infinite cardinal numbers*. There are many infinite sets with different cardinality. We shall exhibit this but will not go deep into development beyond that.

3.6. Definition. Let X be a set. Then the *powerset of X* is the set $\mathscr{P}(X)$ of all subsets of X. Some authors denote it by 2^X also.

Let us recall that for a set X, $\mathscr{P}(X)$ denotes the powerset of X that is, the set of all subsets of X.

3.7. Proposition. *Let X be a set. Then $|X| \leq |\mathscr{P}(X)|$.*

Proof. Indeed $f\colon X \to \mathscr{P}(X)$ defined by $f(x) = \{x\}$, $(x \in X)$, is an injective function from X to $\mathscr{P}(X)$. □

3.8. Proposition. *Let X be a set. There is no function from X onto $\mathscr{P}(X)$.*

Proof. Suppose there is an onto function $f\colon X \to \mathscr{P}(X)$. Define

$$A = \{x \in X : x \notin f(x)\}.$$

Since f is onto, there is $a \in X$ such that $f(a) = A$.

$$\begin{aligned} a \in A &\Rightarrow a \text{ must satisfy the defining condition of } A \\ &\Rightarrow a \notin f(a) = A. \\ a \notin A = f(a) &\Rightarrow a \notin f(a), \text{ which is the defining condition of } A \\ &\Rightarrow a \in A. \end{aligned}$$

That is, $a \in A \Leftrightarrow a \notin A$, an absurdity. Thus f cannot be onto. □

3.9. Corollary. *For a set X, $|X| < \mathscr{P}(X)$.*

3.10. Corollary. *$\mathscr{P}(\mathbb{N})$ is uncountable.*

Now we head towards proving that $\mathbb{R}$ is uncountable.

3.11. Proposition. *A subset of a finite set is finite and a subset of a countable set is countable.*

PROOF. Let X be a finite set and $Y \subset X$. If $Y = \emptyset$, then Y is finite. Suppose $Y \neq \emptyset$. Let $X = \{x_1, x_2, \dots, x_n\}$. Then $Y = \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$ for some $1 \leq i_1 < i_2 < i_3 < \cdots < i_k \leq i_n$, and hence $f\colon Y \to J_{i_k}$, defined by $f(x_{i_j}) = i_j$, $(1 \leq j \leq k)$, is a one to one correspondence. Thus Y is finite.

Let $X = \{x_n : n \in \mathbb{N}\}$ be a countably infinite set and $Y \subset X$. If Y is finite, then we are done. Suppose Y is infinite. Choose the smallest $i_1 \in \mathbb{N}$ such that $x_{i_1} \in Y$. Having chosen $i_1, i_2, \dots, i_k$, choose the smallest $i_{k+1} > i_k$ such that $x_{i_{k+1}} \in Y$. Clearly, $\{x_{i_1}, x_{i_2}, \dots\} \subset Y$. For $x_m \in Y$, let i_p be the largest integer such that $i_p < m$ and $x_{i_p} \in Y$. Then clearly $i_{p+1} = m$. Thus $Y = \{x_{i_1}, x_{i_2}, \dots\}$. Hence $f\colon Y \to \mathbb{N}$, defined by $f(x_{i_j}) = i_j$, $(j \in \mathbb{N})$, is a one-one correspondence. Thus Y is countably infinite. □

3.12. Proposition. $(0, 1)$ *is uncountable.*

PROOF. Suppose $(0, 1)$ is countable and let $f\colon \mathbb{N} \to (0, 1)$ be a one to one correspondence. Define $x_n = f(n)$, $(n \in \mathbb{N})$. Then $(0, 1) = \{x_1, x_2, \dots\}$. Since every element of $(0, 1)$ can be expanded into decimal representation, each x_n can be written as

$$x_n = 0.a_{n1}a_{n2}a_{n3}\dots.$$

Now construct a number $b = 0.b_1b_2b_3\dots$ in the following way. For every $n \in \mathbb{N}$, choose $b_n \in \{1, 2, \dots, 8\} \setminus \{a_{nn}\}$. Then the k^{th} digits of x_k is different from that of b and so, $b \neq x_k$, for every $k \in \mathbb{N}$. A number x has two distinct decimal expansions if and only x has a terminating decimal expansion $x = a.a_1a_2\dots a_j$, $(a_j \geq 1)$, in which case, the second decimal expansion of x is $x = a.a_1a_2\dots a_{j-1}c\bar{9}$, where $c = a_j - 1$ and $\bar{9}$ is considered as the trailing sequence of 9, that is, $999\dots$. Since we have omitted 0 and 9 from the choice of digits of b, there is no possibility of it having repeated 9′s or terminating decimal expansion. Thus b is different from each x_n. Hence $b \in (0, 1) \setminus \{x_1, x_2, \dots\}$. That shows that f is not a one to one correspondence. So, $(0, 1)$ is uncountable. □

3.13. Corollary. $\mathbb{R}$ *is uncountable.*

PROOF. If $\mathbb{R}$ is countable, then by the Proposition 3.11, $(0, 1)$ must be countable, which is not the case. Thus $\mathbb{R}$ is uncountable. □

3.14. Example. A set may be equipotent to it's proper subset. In fact, a subset is in one to one correspondence with some of its proper subset if and only if it is infinite. Indeed let X be an infinite set. Choose a countably infinite subset $A = \{x_1, x_2, \dots\}$ of X. Now define $h\colon X \to X$ by

$$h(x) = \begin{cases} x_{2i}, & \text{if } x = x_i \in A; \\ x, & \text{if } x \in X \setminus A. \end{cases}$$

Then it is easily verified that h is a one to one correspondence between X and $X \setminus \{x_1, x_3, \dots\}$, a proper subset of X.

Conversely, if X is finite, then an application of pigeon hole principle shows that every one-one function from X to X is onto. Thus X cannot be in one to one correspondence with its proper subset.

3.15. Example. $f\colon \mathbb{N} \to 2\mathbb{N} = \{2n : n \in \mathbb{N}\}$ defined by $f(n) = 2n$, $(n \in \mathbb{N})$, is a bijection. As a result, there are as many even natural numbers as there are natural numbers.

3.16. Remark. In view of the Corollary 3.9, we see that $\mathscr{P}(\mathbb{N}), \mathscr{P}(\mathscr{P}(\mathbb{N})), \dots$ is a sequence of uncountable sets having distinct cardinalities. The cardinality of $\mathscr{P}(\mathbb{N})$, denoted by $2^{\aleph_0}$, is the same as the cardinality of $\mathbb{R}$ and it is denoted by $\mathfrak{c}$. Continuum hypothesis states that there is no set having the cardinality strictly between $\aleph_0$ and $2^{\aleph_0} = \mathfrak{c}$.

We have yet not proved a very important property of the cardinal numbers. For sets X, Y if $|X| \le |Y|$ ad $|Y| \le |X|$, then $|X| = |Y|$. This result is known as Schröder–Bernstein theorem. We conclude this section with this theorem.

3.17. Theorem. (Schröder–Bernstein theorem) *For sets X, Y if $|X| \le |Y|$ and $|Y| \le |X|$, then $|X| = |Y|$.*

Proof. Without loss of generality, we assume that X, Y are disjoint sets. Otherwise we label each element $a \in X$ as a_X and each element $a \in Y$ as a_Y and the sets so generated, are disjoint. Since $|X| \le |Y|$ ad $|Y| \le |X|$, there are injective functions $f\colon X \to Y$ and $g\colon Y \to X$. Fix $x \in X$. Let $x_0 = x$. Construct the sequence in $X \cup Y$ as below.

$$\begin{aligned}
y_0 &= f(x_0) & x_1 &= g(y_0) = (g \circ f)(x_0)\\
y_1 &= f(x_1) = (f \circ g)(y_0) & x_2 &= g(y_1) = (g \circ f)^2(x_0)\\
y_2 &= f(x_2) = (f \circ g)^2(y_0) & x_3 &= g(y_2) = (g \circ f)^3(x_0)\\
\dots &= \dots & \dots &= \dots\\
y_{-1} &= g^{-1}(x_0) = (f \circ g)^{-1}(y_0) & x_{-1} &= f^{-1}(y_{-1}) = (g \circ f)^{-1}(x_0)\\
y_{-2} &= g^{-1}(x_{-1}) = (f \circ g)^{-2}(y_0) & x_{-2} &= f^{-1}(y_{-2}) = (g \circ f)^{-2}(x_0)\\
\dots &= \dots & \dots &= \dots
\end{aligned} \tag{3.17.1}$$

First assume that the sequence $x_0, y_0, x_1, x_2, y_2, \dots$ terminates and some $k \ge 0$ is the smallest integer such that x_k repeats. Suppose, if possible, that $k > 0$ and $x_k = x_j$, $j > k$. So, $g(y_{k-1}) = x_k = x_j = g(y_{j-1})$. Since g is injective, $y_{k-1} = y_{j-1}$. Hence $x_{k-1} = f^{-1}(y_{k-1}) = f^{-1}(y_{j-1}) = x_{j-1}$. Since $k > 0$, $k - 1 \ge 0$. Thus k is not the smallest such integer. Hence x_0 must be the first element to repeat. Thus the sequence must be $x_0, y_0, x_1, y_1, \dots, x_n, y_n, x_{n+1} = x_0$, with $x_0, x_1, x_2, \dots, x_n$ distinct elements of X and $y_0, y_1, y_2, \dots, y_n$ distinct elements of Y. Consequently, we have the following types of sequences.

(1) Starting at x_0 and terminating at y_n for some $n \ge 0$.

$x = x_0, y_0, x_1, y_1, \dots, x_n, y_n$.

Define $A_x = \{x = x_0, y_0, x_1, y_1, \dots, x_n, y_n\}$

(2) Infinite on right hand side and terminating on left hand side at x_{-n} for some $n \geq 1$.

$x_{-n}, y_{-n}, \ldots, y_{-1}, x = x_0, y_0, x_1, y_1, \ldots$.

Define $A_x = \{x_{-n}, y_{-n}, \ldots, x = x_0, y_0, x_1, y_1, \ldots\}$

(3) Infinite on right hand side and terminating on left hand side at y_{-n} for some $n \geq 1$.

$y_{-n}, x_{-n+1}, \ldots, y_{-1}, x = x_0, y_0, x_1, y_1, \ldots$.

Define $A_x = \{y_{-n}, x_{-n+1}, \ldots, x = x_0, y_0, x_1, y_1, \ldots\}$

(4) Infinite on both sides.

$\ldots, x_{-n}, y_{-n}, \ldots, x = x_0, y_0, x_1, y_1, \ldots$.

Define $A_x = \{\ldots, x_{-n}, y_{-n}, \ldots, x = x_0, y_0, x_1, y_1, \ldots\}$

We claim that $\{A_x : x \in X\}$ forms a partition of $X \cup Y$. Clearly, $x \in A_x$. Let $y \in Y$. Let $x = g(y)$. Then $y = g^{-1}(x) \in A_x$. Thus $\cup_{x\in X} A_x = X \cap Y$. Also, for $x, z \in X$,

$$\begin{aligned} z \in A_x &\Leftrightarrow s = (g \circ f)^k(x) \quad \text{or } z = (g \circ f)^{-k}(x) \\ &\Leftrightarrow x = (g \circ f)^{-k}(z) \text{ or } x = (g \circ f)^k(x) \\ &\Leftrightarrow x \in A_z \end{aligned}$$

Thus either $A_x = A_z$ or $A_x \cap A_z = \emptyset$. Now define $h\colon X \to Y$ by

$$h(x) = \begin{cases} g^{-1}(x), & \text{if } A_x \text{ terminates on left at } y_{-n} \\ f(x), & \text{otherwise.} \end{cases}$$

Then it is clear that h is a bijection. This completes the proof. □

3.18. Proposition. *Let X be a set and Y be a countable set.*

(1) If there is an injective function from X into Y, then X is countable.
(2) If there is an onto function from Y to X, then X is countable.
(3) Product of two countable sets is countable.
(4) The collection of all finite subsets of a countable set is countable.

Proof. It is sufficient to prove the reslut for infinite sets. Also since an infinite countable set is in one to one correspondence with $\mathbb{N}$, we assume that $Y = \mathbb{N}$.

☞(1) Let $f\colon X \to \mathbb{N}$ be an injective function. Let $f(X) = \{n_1, n_2, n_3, \ldots\}$. Define $g\colon f(X) \to \mathbb{N}$ by $g(n_k) = k$, $(n_k \in f(X))$. Then $g \circ f\colon X \to \mathbb{N}$ is a one to one correspondence. Thus X is countable.

☞ (2) Let $f\colon \mathbb{N} \to X$ be an onto function. For each $x \in f(\mathbb{N}) = X$, choose the smallest integer $n_x \in \mathbb{N}$ such that $f(n_x) = x$. Define $g\colon X \to \mathbb{N}$ by $f(x) = n_x$. Then clearly, f is an injective function from X into $\mathbb{N}$. Thus by (1) X is countable.

☞ (3) Let $A = \{x_1, x_2, x_3, \ldots\}$ and $B = \{y_1, y_2, y_3, \ldots\}$. Define $f\colon A \times B \to \mathbb{N}$ by $f(x_i, y_j) = 2^i 3^j$. Then clearly f is an injective function. Thus by (1) $A \times B$ is countable.

☞ (4) Let $A = \{x_1, x_2, x_3, \dots\}$ and $\mathscr{P}_f(A)$ denote the collection of all finite subsets of A. Let $p_1 < p_2 < p_3 < \dots$ be the listing of all positive prime numbers. Define $h\colon \mathscr{P}_f(A) \to \mathbb{N}$ by $h(\{x_{k_1}, x_{k_2}, x_{k_3}, \dots x_{k_n}\}) = p_1^{k_1} p_2^{k_2} p_3^{k_3} \dots p_n^{k_n}$, $(\{x_{k_1}, x_{k_2}, x_{k_3}, \dots x_{k_n}\} \in \mathscr{P}_f(A))$. Then h is an injective function. This completes the proof by using (1). □

3.19. Corollary. *$\mathbb{Q}$ is countable.*

Problems

11. Prove that the following are equivalent for a set X.

(1) Every surjective $f\colon X \to X$ is injective.
(2) X is finite.
(3) Every injective $f\colon X \to X$ is surjective.

12. In Theorem 3.17, suppose $y \notin f(X)$. Is it true that y is a left end of some sequence, which is infinite on the right hand side?

13. Prove the following.

(1) Subset of an uncountable set need not be uncountable.
(2) A set X is countable if and only if there is a bijection $f\colon \mathbb{N} \to X$ if and only if there is a surjection $g\colon \mathbb{N} \to X$.
(3) Cartesian product as well as union of two countable sets is countable.
(4) Give at least two one to one correspondences between $\mathbb{N}$ and $\mathbb{Z}$.

14. For an infinite set X and countable set Y, show that $|X \cup Y| = |X|$.

4. Set Operations on Arbitrary Families

In school, we are dealing mainly with union, intersection and cartesian product of two or three sets. The concept of these operations is extended to finitely many sets very easily. Even going up to countably infinite sets may not be much difficult. However, here in one go, we generalize the operations for arbitrary family of sets. At times, we may be dealing with many sets. The question arises, "How many?" The answer is, "We do not know while dealing with the problem theoretically. However while in a practical situation we may get to know." If we know the number of sets to be two, we can denote them by A, B or A_1, A_2. If they are three, the choices are A, B, C and A_1, A_2, A_3. However, if there are 100 sets, the natural choice is $A_1, A_2, A_3, \dots, A_{100}$. If we know they are as many as the natural numbers are (*countably infinite*), then usually we write $A_1, A_2, A_3, \dots$. Here we are using $1, 2, 3, \dots$ as the labels of the sets. A label is called an *index*. The labels are called *indices or indexes*. If we know that there are finitely many sets but their exact population is not known, then we list them as $A_1, A_2, A_3, \dots, A_n$. It is understood from the context that $n \in \mathbb{N}$. It may not always be possible to use $1, 2, 3, \dots$ as the indices. Apart from all these situations, there may be uncountably many sets under consideration. In that case, the labels would be coming from some uncountable set. To encompass all this possibilities while maintaining the generality, we must take a generic set of labels. For example, I may be a generic set of labels (not to confuse with an interval) and the labels may be denoted by i, j, k, not to confuse with the integers.

4.1. Definition. Let I be a set, and associated to each $i \in I$, there be a subset A_i of X, a universal set. Then $\{A_i : i \in I\}$ is called an *indexed family* of subsets of X and I is called the *indexing set*.

We start with the definition of the arbitrary union and arbitrary intersection.

4.2. Definition. Let X be a nonempty set and $\{A_i : i \in I\}$ be a family of subsets of X. The *union* of $\{A_i : i \in I\}$ is

$$\cup_{i\in I} A_i = \begin{cases} \{x \in X : x \in A_i \text{ for some } i \in I\}, & \text{if } I \neq \emptyset; \\ \emptyset, & \text{if } I = \emptyset. \end{cases} \tag{4.2.1}$$

4.3. Definition. Let X be a nonempty set and $\{A_i : i \in I\}$ be a family of subsets of X. The *intersection* of $\{A_i : i \in I\}$ is

$$\cap_{i\in I} A_i = \begin{cases} \{x \in X : x \in A_i \text{ for all } i \in I\}, & \text{if } I \neq \emptyset; \\ X, & \text{if } I = \emptyset. \end{cases} \tag{4.3.1}$$

4.4. Remark. Taking different indexing sets I, in Definitions 4.2 and 4.3 all types of unions and intersections are obtained. We say that a union in (4.2.1) and an intersection in (4.3.1) is

(1) *finite*, if I is a finite set. In this case, usually I is taken to be $\{1, 2, 3, \ldots, n\}$ and we write $\cup_{i=1}^{n} A_i$ and $\cup_{i=1}^{n} A_i$.
(2) *countable*, if I is a countable set. Note that this includes a finite union and finite intersection also because a finite set is also a countable set. In this case, usually I is taken to be $\{1, 2, 3, \ldots\}$ and we write $\cup_{i=1}^{\infty} A_i$ and $\cup_{i=1}^{\infty} A_i$.
(3) *arbitrary*, when we do not know a priori if I is finite, countable or uncountable.

Next we turn to the definition of arbitrary product of sets. However, this is not so straightforward. In fact, for years, it could not be resolved if arbitrary product exists or not. Now it is resolved that for some mathematicians it exists for some, it still does not exist. We choose the branch of pure mathematics that accepts the existence of such a product. This is done by accepting the *axiom of choice*. Let us see how this doctrine fits into our brains.

Suppose you are given a box and you are asked to pick up "*an object*" from that box. Can you do that? The answer is yes, provided the box has some object(s) in it before you pick up. Fine, you will agree that if you are given 100 boxes each of which contains some objects, then it is possible for you to choose one object from each of them. Well, now it is simple. If I ask you the same question for 1000 boxes, or 67 boxes, your answer will be the same. What is there in a number? If I give you "*some*" boxes and tell you that each of these boxes contains some objects, then also you can surely choose one object from each of them. It sounds pretty good huh! What if this "*some*" means uncountably many boxes? Well this is the point of dispute(!) among mathematicians. Some mathematicians believe that in any such situation, we should be able to choose one object from each box, whereas

the other group of mathematicians say that before accepting it, we must prove that we can choose one from each.

Reading the above paragraph, one naturally tends to believe that nobody should have any problem in accepting this convincing assumption. However, let us be fair in discussion that it is not so simple. Accepting the above one can prove mathematically that every set can be well-ordered (see Defintion 6.23). That is, say for example, we can arrange all elements of $\mathbb{R}$ in such a way that every nonempty subset of $\mathbb{R}$ has the *first element*. Now some of the readers will deviate from accepting the Axiom of Choice. Because the Axiom of Choice is convincing but not it's corollary. Well, the debate continues. However, let us also note that accepting the AC or not accepting the AC will generate a consistent system in mathematics. So, one is free to choose one's own conviction. We accept AC.

4.5. Theorem. (Axiom of Choice) *Let $\{A_i : i \in I\}$ be a nonempty family of nonempty sets. Then there exists a function $f : I \to \cup_{i\in I} A_i$ such that $f(i) \in A_i$.*

Note that the existence of such a function is the same as selecting one object from each A_i. In other words, if $\{A_i : i \in I\}$ is a nonempty family of nonempty pairwise disjoint sets, then there exists a set A such that $A \cap A_i$ is singleton for every $i \in I$.

The following definition gives a nonempty product of arbitrary family of nonempty sets only because of the Axiom of Choice.

4.6. Definition. For a family $\{X_i : i \in I\}$ of sets, the *cartesian product of* $\{X_i : i \in I\}$ is defined as,

$$\prod_{i\in I} X_i = \begin{cases} \{\mathbf{x} : I \to \cup X_i : \mathbf{x}_i \in X_i \text{ for each } i \in I\}, & \text{if } I \neq \emptyset \text{ and each } X_i \neq \emptyset; \\ \emptyset, & \text{if } I = \emptyset \text{ or } X_i = \emptyset \text{ for some } i. \end{cases} \tag{4.6.1}$$

II. N$^o{}_t$ations: ☞

(1) $X^I = \prod_{i\in I} X_i$, where $X_i = X$ for each $i \in I$.
(2) $X^n = \prod_{i=1}^{n} X_i$, where $X_i = X$ for each $1 \leq i \leq n$.
(3) $X^{\mathbb{N}} = \prod_{i=1}^{\infty} X_i$, where $X_i = X$ for each $1 \leq i < \infty$.

4.7. Remark.

(1) Each X_i is called a *coordinate space* of $\prod_{i\in I} X_i$.
(2) Though an element $\mathbf{x} \in \prod_{i\in I} X_i$ is a function, its value $\mathbf{x}(i)$ at i is written as $\mathbf{x}_i$ and not $\mathbf{x}(i)$. The element $\mathbf{x}$ is written as $\mathbf{x} = (\mathbf{x}_i)_{i\in I}$ or for brevity, $\mathbf{x} = (\mathbf{x}_i)$. $\mathbf{x}_i$ is called *i^{th} coordinate* of $\mathbf{x}$.
(3) By taking different indexing sets I, all types of products are obtained as particular cases. We say that (4.6.1) is a

(a) *finite product*, if I is a finite set. In this case, usually I is taken to be $\{1, 2, 3, \dots, n\}$ and $\prod_{i\in I} X_i$ is written as $\prod_{i=1}^{n} X_i$.
(b) *countable product*, if I is a countable set. Note that this includes a finite product also because a finite set is also a countable set. In case, I is countably infinite, usually I is taken to be $\{1, 2, 3, \dots\}$ and $\prod_{i\in I} X_i$ is written as $\prod_{i=1}^{\infty} X_i$.
(c) *arbitrary product*, if we do not know the cardinality of I.

(4) One can easily see that $X^{\mathbb{N}}$ is the set of all sequences in X.
(5) Since the elements of a product space are realized as a function, there is no importance of the order on the indexing sets. Thus $X_1 \times X_2$ is the same as $X_2 \times X_1$. More generally, for a partition $\{I_\alpha : \alpha \in \Gamma\}$ of the indexing set I, $\prod_{i\in I} X_i = \prod_{\alpha\in\Gamma} \prod_{i_\alpha \in I_\alpha} X_{i_\alpha}$.

4.8. Definition. Let $\{X_i : i \in I\}$ be a family of sets and let $X = \prod_{i\in I} X_i$. For $j \in I$, we define the *j^{th}-projection map* $\pi_j \colon X \to X_j$ by $\pi_j(\mathbf{x}) = \mathbf{x}_j$, $(\mathbf{x} = (\mathbf{x}_i) \in X)$. The map is also called *j^{th}-coordinate map.*

4.9. Example. Let $I = [-1, 2]$ and for each $i \in I$, let $X_i = \mathbb{R}$. Here are some elements of X^I.

$\mathbf{x}_t = \frac{9}{100} t \sin(\pi t)$.

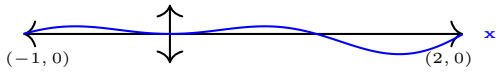


FIGURE 1. $\mathbf{x} = (\mathbf{x}_t)$

$$\mathbf{w}_t = \begin{cases} 1.2, & \text{if } t \in [-1, 0] \cap \mathbb{Q}; \\ -t, & \text{if } t \in [-1, 0] \cap (\mathbb{R} \cap \mathbb{Q}); \\ \frac{5t+9}{10}, & \text{if } t \in [0, 2] \cap \mathbb{Q}; \\ (\frac{t}{2})^2 + \frac{3}{10}, & \text{if} t \in [0, 2] \cap (\mathbb{R} \cap \mathbb{Q}). \end{cases}$$

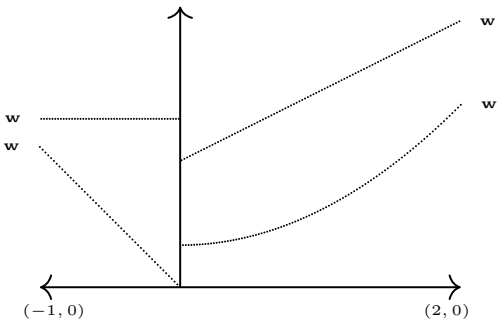


FIGURE 2. $\mathbf{w} = (\mathbf{w}_t)$

$\mathbf{y}_t = \frac{e^{sin(\pi t)}+1}{5}$.

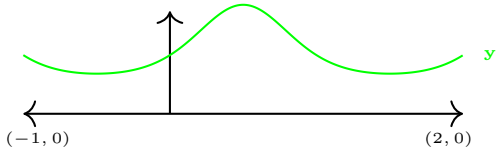


FIGURE 3. $\mathbf{y} = (\mathbf{y}_t)$

$$\mathbf{z}_t = \begin{cases} -\frac{t}{5}, & \text{if } t \in [-1,0]; \\ \frac{3t+2}{10}, & \text{if } t \in [0,1]; \\ -\frac{t^2}{10}, & \text{if } t \in [1,2]. \end{cases}$$

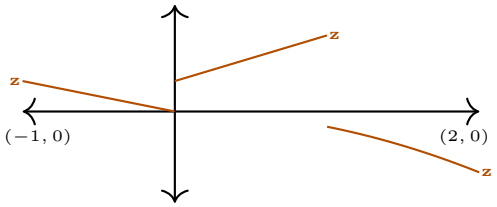


FIGURE 4. $\mathbf{z} = (\mathbf{z}_t)$

4.10. Lemma. *Let $\{X_i : i \in I\}$ be a family of sets, $X = \prod_{i\in I} X_i$ and $j, k \in I$. Also let $A_i, B_i \subset X_i$ for all $i \in I$.*

(1) A projection is an onto map.
(2) $\cap_{i\in I}\pi_i^{-1}(A_i) = \prod_{i\in I} A_i$.
(3) $\pi_j^{-1}(A_j) = A_j \times \prod_{\substack{i\in I \\ i\neq j}} X_i$.
(4) $(\prod_{i\in I} A_i) \cap (\prod_{i\in I} B_i) = \prod_{i\in I}(A_i \cap B_i)$.
(5) $\pi_j(\pi_j^{-1}(A_j)) = A_j$.
(6) If $A_j \neq \emptyset$ and $j \neq k$, then $\pi_k(\pi_j^{-1}(A_j)) = X_k$.
(7) If for every $i \in I$, $A_i \neq \emptyset$, then $\pi_j(\prod_{i\in I} A_i) = A_j$.

PROOF. ☞(1) Let $\mathbf{x} = (\mathbf{x}_i) \in X$. FIX $j \in I$ and $a \in X_j$. Define $\mathbf{y} \in X$ by

$$\mathbf{y}_i = \begin{cases} \mathbf{x}_i, & \text{if } i \neq j; \\ a, & \text{if } i = j. \end{cases}$$

That is, $\{\mathbf{y}\} = \{a\} \times \prod_{\substack{i\in I \\ i\neq j}} \{\mathbf{x}_i\}$. But then $\pi_j(\mathbf{y}) = a$. Thus π_j is onto.

☞(2)
$$\begin{aligned} \mathbf{x} \in \cap_{i\in I}\pi_i^{-1}(A_i) &\Leftrightarrow \mathbf{x} \in \pi_i^{-1}(A_i) \text{ for every } i \in I \\ &\Leftrightarrow \pi_i(\mathbf{x}) \in A_i \text{ for every } i \in I \\ &\Leftrightarrow \mathbf{x}_i \in A_i \text{ for every } i \in I \\ &\Leftrightarrow \mathbf{x} \in \prod_{i\in I} A_i. \end{aligned}$$

☞(3) Using (2), $A_j \times \prod_{\substack{i\in I \\ i\neq j}} X_i = \pi_j^{-1}(A_j) \cap (\cap_{\substack{i\in I \\ i\neq j}} \pi_i^{-1}(X_i)) = \pi_j^{-1}(A_j \cap X) = \pi_j^{-1}(A_j)$ because $\pi_j^{-1}(X_i) = X$.

☞(4)
$$\begin{aligned} \left(\prod_{i\in I} A_i\right) \cap \left(\prod_{i\in I} B_i\right) &= (\cap_{i\in I}\pi_i^{-1}(A_i)) \cap (\cap_{i\in I}\pi_i^{-1}(B_i)) \\ &= \cap_{i\in I}(\pi_i^{-1}(A_i) \cap \pi_i^{-1}(B_i)) \\ &= \cap_{i\in I}(\pi_i^{-1}(A_i \cap B_i)) \\ &= \prod_{i\in I}(A_i \cap B_i). \end{aligned}$$

☞(5) follows because π_j is onto.

☞(6) One way proof is xobvious. On the other hand, let $a \in X_k$. Fix $\mathbf{y} \in \{a\} \times A_j \times \prod_{\substack{i\in i \\ j\neq i\neq k}} X_i$. Then $\mathbf{x} \in \pi_j^{-1}(A_j)$ and $\pi_k(\mathbf{x}) = a$. This completes the proof.

☞(7) Let $a \in \pi_j(\prod_{i\in I} A_i)$. So, there is $\mathbf{x} \in \prod_{i\in I} A_i$ such that $\mathbf{x}_j = \pi_j(\mathbf{x}) = a$. But $\mathbf{x} \in \prod_{i\in I} A_i \Rightarrow \mathbf{x}_j \in A_j$. Thus $a = \mathbf{x}_j \in A_j$.

Conversely, let $a \in A_j$. Since each $A_i \neq \emptyset$, choose $\mathbf{x} \in \{a\} \times \prod_{\substack{i\in I \\ i\neq j}} A_i$ Then $a = \pi_j(\mathbf{x}) \in \pi_j(\prod_{i\in I} A_i)$. □

4.11. Definition. For a family $\{X_i : i \in I\}$ of sets, let $X = \prod_{i\in I} X_i$ and Y be a set. For $f\colon Y \to X$, the maps $f_i = \pi_i \circ f$, $(i \in I)$, are called the *coordinate maps* of f.

The following gives a lifting of coordinate maps to construct f out of f_i.

4.12. Theorem. *For a family $\{X_i : i \in I\}$ of sets, let $X = \prod_{i\in I} X_i$ and Y be a set. Suppose for each $i \in I$, there is a map $f_i\colon Y \to X_i$. Then there is a unique $f\colon Y \to X$ such that f_i are coordinate maps of f, that is $f_i = \pi_i \circ f$ for each $i \in I$. In other words $f(x)_i = f_i(x)$ for each $i \in I$.*

Proof. For $y \in Y$, define $f(y)_i = f_i(y)$, $(i \in I)$. Then $f\colon Y \to X$ is a function. Also, for $i \in I$, $\pi_i \circ f(y) = f(y)_i = f_i(y)$ for all $y \in Y$. Thus $\pi_i \circ f = f_i$. Uniqueness is straight forward. □

4.13. Definition. For a family $\{X_i : i \in I\}$ of sets, let $X = \prod_{i\in I} X_i$ and Y be a set. Suppose for each $i \in I$, there is a map $f_i\colon Y \to X_i$. Then the unique map f in the above theorem is called the *lifting* of f_i or *evaluation map*.

4.14. Definition. Let $\{X_i : i \in I\}$ be a family of sets, $X = \prod_{i\in I} X_i$ and Y be a set. Suppose for each $i \in I$, there is a map $f_i\colon Y \to X_i$. Then the family $\{f_i : i \in I\}$ is said to *separate points* of Y if for every distinct $y, z \in Y$, there is $i \in I$ such that $f_i(y) \neq f_i(z)$.

4.15. Theorem. *Let $\{X_i : i \in I\}$ be a family of sets and let $X = \prod_{i\in I} X_i$ and Y be a set. Suppose for each $i \in I$, there is a map $f_i\colon Y \to X_i$. Then the lifting is one-one if and only if $\{f_i : i \in I\}$ separates the points of Y.*

Proof. Let $y, z \in Y$. Then

$$\begin{aligned} f \text{ is injective} &\Leftrightarrow f(y) \neq f(z) \\ &\Leftrightarrow f(y)_i \neq f(z)_i \text{ for some } i \in I \\ &\Leftrightarrow f_i(y) \neq f_i(z) \text{ for some } i \in I \\ &\Leftrightarrow \{f_i : i \in I\} \text{ separates } y \text{ and } z. \end{aligned}$$

□

4.16. Definition. Let $\{X_i : i \in I\}$ be a family of sets. The set

$$\coprod_{i\in I} X_i = \{(x, i) : i \in I, x \in X_i\} \tag{4.16.1}$$

is called the *disjoint union* or *coproduct* of $\{X_i : i \in I\}$. In this case, we define $X_i^* = \{(x, i) : x \in X_i\}$ and for each $i \in I$, the injection $\iota_i \colon X_i \to \coprod_{i\in I} X_i$ by $\iota_i(x) = (x, i)$, $(x \in X_i)$.

III. N$^o{}_t$ations: ☞ For $j \in I$ and $B \subset X_j$, we set $B \coprod \emptyset = \coprod_{i\in I} B_i$, with $B_i = \begin{cases} B, & \text{if } i = j; \\ \emptyset, & \text{otherwise.} \end{cases}$

Problems

15. Find $\cap_{n=1}^{\infty} A_n$ and $\cup_{n=1}^{\infty} A_n$, for the following.

(1) $A_n = (0, \frac{1}{n})$, $n \in \mathbb{N}$.
(2) $A_n = [0, \frac{1}{n}]$, $n \in \mathbb{N}$.
(3) $A_n = (-\frac{1}{n}, \frac{1}{n})$, $n \in \mathbb{N}$.
(4) $A_n = [-\frac{1}{n}, \frac{1}{n}]$, $n \in \mathbb{N}$.
(5) $A_n = (0, n)$, $n \in \mathbb{N}$.
(6) $A_n = [n, n + 1)$, $n \in \mathbb{N}$.
(7) $A_n = (n, n + 1]$, $n \in \mathbb{N}$.
(8) $A_n = (n, \infty)$, $n \in \mathbb{N}$.

16. For $x \in \mathbb{R}$, let $A_x = [x, x + 1]$. Find $\cup_{x\in[0,\frac{1}{4}]} A_x$, $\cup_{x\in[0,2]} A_x$, $\cap_{x\in[0,\frac{1}{4}]} A_x$ and $\cap_{x\in[0,2]} A_x$.

17. Let $\{A_i : i \in J\}$ be a family of subsets of X. Find the relations between the sets $\cup_{i\in I} A_i$, $\cup_{i\in J} A_i$, $\cap_{i\in I} A_i$ and $\cap_{i\in J} A_i$ under each of the assumptions, $\emptyset \neq I \subsetneqq J$ and $\emptyset = I \subsetneqq J$.

18. Let $\{A_i : i \in I\}$ be a family of subsets of X. Prove the following.

(1) $\cup_{i\in I}(X \smallsetminus A_i) = X \smallsetminus \cap_{i\in I} A_i$.[2]
(2) $\cap_{i\in I}(X \smallsetminus A_i) = X \smallsetminus \cup_{i\in I} A_i$.[2]

19. Can we express $(0, 1)$ as an intersection of open intervals each of which has left end point other than 0?

20. Prove or disprove: $\cap_{n=5}^{\infty}(1 - \frac{1}{n}, 1 + \frac{1}{n}) = \cap_{n=5}^{\infty}[1 - \frac{1}{n}, 1 + \frac{1}{n}]$.

21. Prove or disprove: $\cup_{n=5}^{\infty}(1 + \frac{1}{n}, 1 - \frac{1}{n}) = \cup_{n=5}^{\infty}[1 + \frac{1}{n}, 1 - \frac{1}{n}]$.

22. Prove or disprove: $(0, 1)$ is a union of closed intervals.

23. Prove or disprove: $[0, 1]$ as an intersection of open intervals.

24. Show that a set X is infinite if and only if there is a family $\mathscr{F} = \{A_i : i \in I\}$ of subsets of X such that intersection of finitely many members of $\mathscr{F}$ is nonempty but $\cap\mathscr{F} = \emptyset$.

25. Let X, Y be two sets and R be a relation from X to Y. For a family $\{A_i : i \in I\}$ of subsets of X, show that

(1) $R(\cup_{i\in I} A_i) = \cup_{i\in I} R(A_i)$.

[2] This is called De Morgan's Law

(2) $R(\cap_{i\in I} A_i) \subset \cap_{i\in I} R(A_i)$. Characterize the equality. Suppose, $x_1, x_2 \in X, x_1 \neq x_2 \Rightarrow R(x_1) \cap R(x_2) = \emptyset$. Does the equality hold?

26. For a family $\{X_i : i \in I\}$ of sets and $A_i, B_i \subset X_i$ for all $i \in I$, prove that $(\prod_{i\in I} A_i) \cup (\prod_{i\in I} B_i) \subset \prod_{i\in I}(A_i \cup B_i)$. Also, show that the equality need not hold.

27. Consider a product $X = \prod_{i\in I} X_i$. Show that the following situations are possible. Are they the only situations? Explain.

(1) Exactly one projection is one-one.
(2) All projections are one-one.
(3) No projection is one-one.

28. Can we have a product of five sets such that exactly two projections are one-one?

5. Equivalence Relation and Quotient

5.1. Definition. Let X be a nonempty set. A relation R on X is said to be

(1) *reflexive* if $\Delta \subset R$. That is, $x \in X \Rightarrow xRx$.
(2) *symmetric* if $R = R^{-1}$. That is, $xRy \Rightarrow yRx$.
(3) *antisymmetric* if $R \cap R^{-1} \subset \Delta$. That is, $xRy, yRx \Rightarrow x = y$.
(4) *transitive* if $R \circ R \subset R$. That is, $xRy, yRz \Rightarrow xRz$.
(5) *equivalence* if R is reflexive, symmetric and transitive.
(6) *antireflexive* if $R \cap \Delta = \emptyset$.

5.2. Remark. Usually $\sim, \simeq, \cong$ or $\equiv$ is used to denote an equivalence relation.

5.3. Example. Let X, Y be two nonempty sets and $f\colon X \to Y$ be a function. Define, for $x, y \in X$, $x \sim y$ if $f(x) = f(y)$. Then $\sim$ is an equivalence relation.

5.4. Example. For $n, m \in \mathbb{Z}$, define $m \equiv n \mod 7$ if $7 \mid (m - n)$. Then $\equiv \mod 7$ is an equivalence relation.

5.5. Example. For $z, w \in \mathbb{C}$, define $z \sim w$ if $z = e^{i\theta}w$ for some $\theta \in \mathbb{R}$. This is as good as saying that $z \sim w$ if $|z| = |w|$. Then $\sim$ is an equivalence relation.

FIGURE 5. Reflexive Relation

FIGURE 6. Symmetric Relation

FIGURE 7. Transitive Relation

5.6. Definition. For an equivalence relation $\sim$ on a nonempty set X and $x \in X$, the set

$$[x] = \{y \in X : x \sim y\}.$$

is called the *equivalence class of $\sim$ containing x*.

5.7. Definition. Let X be a nonempty set. A collection $\{A_i : i \in I\}$ of subsets of X is said to be

(1) *pairwise disjoint* if for $i, j \in I$, either $A_i = A_j$ or $A_i \cap A_j = \emptyset$.
(2) *partition* of X if it is pairwise disjoint and $\cup A_i = X$.

5.8. Theorem. *Let X be a nonempty set and $\sim$ be an equivalence relation on X. Then the equivalence classes of $\sim$ form a partition of X.*

Proof. Let $a \in X$. Then $a \in [a] \subset \cup\{[x] : x \in X\}$. Thus $X \subset \cup\{[x] : x \in X\}$. Now suppose for $x, y \in X$, $a \in [x] \cap [y]$. Consequently,

$$\begin{aligned}
b \in [x] &\Rightarrow x \sim b \\
&\Rightarrow x \sim b, x \sim a, y \sim a && \text{(above combined with } a \in [x], a \in [y]) \\
&\Rightarrow b \sim x, x \sim a, a \sim y && \text{(because } \sim \text{ is symmetric)} \\
&\Rightarrow b \sim a, a \sim y && \text{(because } \sim \text{ is transitive)} \\
&\Rightarrow b \sim y && \text{(because } \sim \text{ is transitive)} \\
&\Rightarrow y \sim b && \text{(because } \sim \text{ is symmetric)} \\
&\Rightarrow b \in [y].
\end{aligned}$$

Thus $[x] \subset [y]$.

Interchanging x, y, it follows that $[y] \subset [x]$. Thus $[x] = [y]$. Thus either $[x] \cap [y] = \emptyset$ or $[x] = [y]$. Hence equivalence classes of $\sim$ form a partition of X. □

5.9. Definition. Let X be a nonempty set and $\sim$ be an equivalence relation on X. The partition $\{[x] : x \in X\}$ of X is called the *quotient of X by $\sim$* and is denoted by $X/\sim$ (read "*X quotient tilde*").

5.10. Theorem. *Let X be a nonempty set and $\mathfrak{F} = \{A_i : i \in I\}$ be a partition of X. Define for $x, y \in X$, $x \sim y$ if there is $i \in I$ such that $x, y \in A_i$. Then $\sim$ is an equivalence relation on X and $X/\sim = \mathfrak{F}$.*

Proof. Let $x \in X$. Since $\cup A_i = X$, there is $i \in I$ such that $x \in A_i$, that is, $x, x \in A_i$. Thus $x \sim x$. So, $\sim$ is reflexive. Now suppose that $x, y \in X$ and $x \sim y$. So, there is $i \in I$ such that $x, y \in A_i$. But then $y, x \in A_i$ gives $y \sim x$. Thus $\sim$ is symmetric. Finally, suppose $x, y, z \in X$, $x \sim y$ and $y \sim z$. Hence there are $i, j \in I$ such that $x, y \in A_i$ and $y, z \in A_j$. But then $y \in A_i \cap A_j$. Consequently, $A_i = A_j$. This, in turn, gives $x, y, z \in A_i$. Hence $x \sim z$. So, $\sim$ is transitive. □

The following is found under the name of "the First Isomorphism Theorem" in various courses like group theory, ring theory, linear algebra, Banach algebras etc.

5.11. Theorem. *Let X, Y be two sets and $f\colon X \to Y$ be an onto function. Define $\sim_f$ on X by $x \sim_f y$ if $f(x) = f(y)$, $(x, y \in X)$. Define $\bar{f}\colon X/\sim_f \to Y$ by $\bar{f}([x]) = f(x)$, $([x] \in X/\sim_f)$. Then $\bar{f}$ is a one to one correspondence between $X/\sim_f$ and Y.*

Proof. It is clear that

$$[x] = [y] \Leftrightarrow x \sim_f y \Leftrightarrow f(x) = f(y) \Leftrightarrow \bar{f}([x]) = \bar{f}[y].$$

The *if part* shows that $\bar{f}$ is well-defined and the *only if* part shows that $\bar{f}$ is one-one. Also, for $y \in Y$, there is $x \in X$ such that $f(x) = y$. So, $\bar{f}[x] = y$, establishing that $\bar{f}$ is onto. □

5.12. Definition. Let X, Y be two sets and $f\colon X \to Y$ be an onto function. For $x \in X$, let $[x]$ denote the equivalence class of x defined by the equivalence relation $\sim_f$. The set $X/\sim_f$ is called *the quotient of X by f* and the function $q_f\colon X \to X/\sim_f$ defined by $q_f(x) = [x]$, $(x \in X)$, is called the *quotient map*. Then the function $\bar{f}\colon X/\sim_f \to Y$ is called *factor of f* and f is said to factor through $X/\sim_f$. A set so obtained by any function defined on the set X is called a *quotient of X*.

5.13. Remark. It is customary to write X/f for $X/\sim_f$. We shall use this in the Section 21 while discussing quotient spaces of topological spaces.

5.14. Example. For $x, y \in \mathbb{R}$ define $x \sim y$ if $x - y \in \mathbb{Z}$. Then $\sim$ is an equivalence relation and $|X/\sim| = |[0, 1)|$. Indeed the proof of equivalence follows from the following three observations.

(1) $x - x = 0 \in \mathbb{Z}$ for all $x \in \mathbb{R}$.
(2) $x - y \in \mathbb{Z} \Rightarrow y - x \in \mathbb{Z}$.
(3) $x - y, y - z \in \mathbb{Z} \Rightarrow x - z = x - y + y - z \in \mathbb{Z}$.

For a real number x let $\lfloor x \rfloor$, read as *floor x*, be the greatest integer n such that $n \leq x$. Now define $f\colon \mathbb{R} \to [0, 1)$ by $f(x) = x - \lfloor x \rfloor$, $(x \in \mathbb{R})$. then the reader needs to prove the following to establish $|X/\sim| = |[0, 1)|$. We leave the verification to the reader.

(1) For every $x \in \mathbb{R}$, $0 \leq x - \lfloor x \rfloor < 1$.
(2) $f(x) = f(y)$ if and only if $x \sim y$.
(3) $f_{|_{[0,1)}}$ is the identity map, proving that f is onto.

5.15. Example. Let us identify the two end points of the interval $[0, 1]$ to get the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. This is obtained by the function $f\colon [0, 1] \to S^1$ defined by $f(t) = e^{2\pi i t}$, $(t \in [0, 1])$. Thus $S^1 = [0, 1]/f$

Problems

29. Show that intersection of two equivalence relations on a set X is again an equivalence relation on X.

30. Let X be a set with n elements. Find the

(1) number of relations on X.
(2) number of antireflexive relations on X.
(3) number or reflexive relations on X.
(4) symmetric antireflexive relations on X.
(5) symmetric relations on X.

31. Show that composition of two equivalence relations on a set X need not be an equivalence relation on X.

32. For a nonempty set X, show that Δ is an equivalence relation on X.

33. Give an example of a relation which is

(1) symmetric and transitive but not reflexive.
(2) reflexive and symmetric but not transitive.
(3) reflexive and transitive but not symmetric.
(4) reflexive and antisymmetry but not transitive.

34. Classify the following relations into reflexive, symmetric and transitive.

(1) On $\mathbb{N}$, nRm if $\min\{n, m\} < |n - m|$, $(n, m \in \mathbb{N})$.
(2) On $\mathbb{R}$, xRy if $x - y \in \mathbb{Q}$, $(x, y \in \mathbb{R})$.
(3) On $\mathbb{R}$, xRy if $x + y \in \mathbb{Q}$, $(x, y \in \mathbb{R})$.
(4) On $\mathbb{R}$, xRy if xy is a rational number, $(x, y \in \mathbb{R})$.
(5) On $\mathbb{R} \setminus \{0\}$, xRy if $\frac{x}{y}$ is a rational number, $(x, y \in \mathbb{R} \setminus \{0\})$.
(6) On $\mathbb{C} \setminus \{0\}$, zRw if $|zw^{-1}| = 1$, $(z, w \in \mathbb{C})$.

35. Describe equivalence classes for the following relations.

(1) For $x, y \in \mathbb{R}$, $x \sim y$ if $x - y \in \mathbb{Z}$.
(2) For $x, y \in \mathbb{R}$, $x \sim y$ if $x - y \in \mathbb{Q}$.
(3) For $x, y \in \mathbb{Z}$, $x \sim y$ if $2 \mid (x - y)$.
(4) For $x, y \in \mathbb{R}$, $x \sim y$ if $|x| = |y|$.
(5) For $x, y \in \mathbb{R}$, $x \sim y$ if $x^2 = y^2$.
(6) For $x, y \in \mathbb{C}$, $x \sim y$ if $|x| = |y|$.
(7) For $p(x), q(x) \in \mathbb{R}[x]$, $p(x) \sim q(x)$ if $p(t) = q(t)$ for $t \in \{1, \frac{1}{2}, \frac{1}{3}\}$.
(8) For $p(x), q(x) \in \mathbb{R}[x]$, $p(x) \sim q(x)$ if $p(t) = q(t)$ for $t \in 1, \frac{1}{2}, \frac{1}{3}, \ldots$.
(9) For polynomials $p(x), q(x) \in \mathbb{R}[x]$, $p(x) \sim q(x)$ if $p(t) = 0 \Leftrightarrow q(t) = 0$, $(t \in \mathbb{R})$.

6. Order

An *order* is a special type of relation. The order theory is vast. Covering even iota of it is out of the scope of the present notes. We shall include the working knowledge required for topology.

6.1. Definition. Let X be a nonempty set and R be a relation on X. R is said to be a

(1) *partial order* if R is reflexive, antisymmetric and transitive. Usually a partial order is denoted by $\leq$.
(2) *total order* or a *linear order* if it is a partial order and for $x, y \in X$, either xRy or yRx.
(3) *strict order* if $R = S \setminus \Delta$ for some partial order S. Usually a strict order is denoted by $<$. Depending upon whether S is a partial or a total order, R is called a *strict partial order* or a *strict total order*.

A set X with a partial order $\leq$ is called a *poset* or a *partially ordered set* and is denoted by $(X, \leq)$. A set with a linear order is called a *linearly ordered set* or a *totally ordered set* or a *chain*.

6.2. Remark.

(1) It is customary to write $\geq$ for $\leq^{-1}$ and $>$ for $<^{-1}$.
(2) On a set X, $\leq$ is a partial order (respectively, a total order) if and only if so is $\geq$.
(3) On a set $<$ is a strict order if and only if so is $>$.
(4) For a strict order $<$ on a set, $< \cup \Delta$ is a partial order and for a partial order $\leq$ the order $\leq \setminus \Delta$ is a strict order.
(5) Let $(X, \leq)$ be a poset. The elements $x, y \in X$ are said to be *comparable*, if either $x_1 \leq x_2$ or $x_2 \leq x_1$.

6.3. Example. Let X be a nonempty set. Then Δ is a partial order on X. In fact, Δ is the only relation on X which is equivalence as well as a partial order.

6.4. Example. Let X be a nonempty set. For $A, B \in \mathscr{P}(X)$, define $A \leq B$ if $A \subset B$.[3] Associated to this relation are three other relations. (1) $\subsetneq$ the strict order. (2) $\supset$ the reversed order. (3) $\supsetneq$ the reversed strict order. All the four of these are total on $\mathscr{P}(X)$ if and only if X has at most one element.

6.5. Example. On $\mathbb{N}$, define $n \leq m$ if $n \mid m$. The following are easy to observe.

(1) $n \mid n$ for every $n \in \mathbb{N}$.
(2) $n \mid m, m \mid n \Rightarrow m = n$.
(3) $n \mid m, m \mid p \Rightarrow n \mid p$.

Thus $\mathbb{N}$ is a poset with the divisibility relation.

6.6. Example. We if we define the above relation on $\mathbb{Z}$, that is, on $\mathbb{Z}$, define $n \leq m$ if $n \mid m$. Then the antisymmetry fails because $4 \mid -4$ and $-4 \mid 4$ but $4 \neq -4$.

6.7. Example. Let $(X, \leq)$ be a poset. Then every nonempty subset of X with the order inherited from X is a poset.

6.8. Definition. Let $(X, \leq_X), (Y, \leq_Y)$ be two posets. An bijective $f\colon X \to Y$ is said to be *order isomorphism* if $a \leq_X b \Leftrightarrow f(a) \leq_Y f(b)$, $(a, b \in X)$. In this case, we say that $(X, \leq_X)$ and $(Y, \leq_Y)$ have the same *order type*.

[3] Recall that $A \subset B$ also includes the possibility of equality. Thus this is not a strict order.

6.9. Proposition. *Let X_1, X_2 be posets. For $(\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2) \in X_1 \times X_2$, define $(\mathbf{x}_1, \mathbf{x}_2) \leq (\mathbf{y}_1, \mathbf{y}_2)$ if either $\mathbf{x}_1 < \mathbf{y}_1$ or $\mathbf{x}_1 = \mathbf{y}_1$ and $\mathbf{x}_2 \leq \mathbf{y}_2$. Then $\leq$ is a partial order on $X_1 \times X_2$.*

PROOF. For every $(\mathbf{x}_1, \mathbf{x}_2) \in X_1 \times X_2$, clearly, $(\mathbf{x}_1, \mathbf{x}_2) \leq (\mathbf{x}_1, \mathbf{x}_2)$ because $\mathbf{x}_1 = \mathbf{x}_1$ and $\mathbf{x}_2 \leq \mathbf{x}_2$. Now for $(\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2) \in X_1 \times X_2$, assume that $(\mathbf{x}_1, \mathbf{x}_2) \leq (\mathbf{y}_1, \mathbf{y}_2)$ and $(\mathbf{y}_1, \mathbf{y}_2) \leq (\mathbf{x}_1, \mathbf{x}_2)$. If $\mathbf{y}_1 < \mathbf{x}_1$, then $(\mathbf{x}_1, \mathbf{x}_2) \not\leq (\mathbf{y}_1, \mathbf{y}_2)$. Thus $\mathbf{x}_1 \leq \mathbf{y}_1$. Similarly, $\mathbf{y}_1 \leq \mathbf{x}_1$. Thus $\mathbf{x}_1 = \mathbf{y}_1$. The same argument on $\mathbf{x}_2, \mathbf{y}_2$ yields to $\mathbf{x}_2 = \mathbf{y}_2$. Let $(\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2), (\mathbf{z}_1, \mathbf{z}_2) \in X_1 \times X_2$. with $(\mathbf{x}_1, \mathbf{x}_2) \leq (\mathbf{y}_1, \mathbf{y}_2)$ and $(\mathbf{y}_1, \mathbf{y}_2) \leq (\mathbf{z}_1, \mathbf{z}_2)$. If either $\mathbf{x}_1 < \mathbf{y}_1$ or $\mathbf{y}_1 < \mathbf{z}_1$, then $\mathbf{x}_1 < \mathbf{z}_1$, and hence, we are done. So, assume that $\mathbf{x}_1 = \mathbf{y}_1 = \mathbf{z}_1$. But then the order between the three ordered pairs implies that $\mathbf{x}_2 \leq \mathbf{y}_2$ and $\mathbf{y}_2 \leq \mathbf{z}_2$, which, in turn, yields to $\mathbf{x}_2 \leq \mathbf{z}_2$. Thus $(\mathbf{x}_1, \mathbf{x}_2) \leq (\mathbf{z}_1, \mathbf{z}_2)$. This completes the proof. □

6.10. Definition. Let $(X_1, \leq), (X_2, \leq)$ be posets. Then the order defined in Proposition 6.9 is called the *lexical order* or the *lexicographic order* or the *dictionary order*.

6.11. Corollary. *Let $(X_1, \leq), (X_2, \leq)$ be posets, and $X_1 \times X_2$ carry the lexicographic order. Then $X_1 \times X_2$ is linearly ordered if and only if $(X_1, \leq)$ and $(X_2, \leq)$ are linearly ordered.*

PROOF. Suppose $X_1 \times X_2$ is linearly ordered. Let $a, b \in X_1$. Choose $c \in X_2$. Then $(a, c), (b, c)$ are comparable. Without loss of generality, assume that $(a, c) \leq (b, c)$. But then either $a \leq b$ or $a = b$. Thus a, b are comparable. The similar argument proves that X_2 is linearly ordered too.

Conversely, suppose X_1, X_2 both are linearly ordered. Let $(\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2) \in X_1 \times X_2$. If $\mathbf{x}_1 < \mathbf{y}_1$ or $\mathbf{y}_1 < \mathbf{x}_1$, then accordingly, $(\mathbf{x}_1, \mathbf{x}_2) \leq (\mathbf{y}_1, \mathbf{y}_2)$ or $(\mathbf{y}_1, \mathbf{y}_2) \leq (\mathbf{x}_1, \mathbf{x}_2)$. Otherwise, the comparability of $\mathbf{x}_1, \mathbf{y}_1$ implies $\mathbf{x}_1 = \mathbf{y}_1$. In that case depending upon $\mathbf{x}_2 \leq \mathbf{y}_2$ or $\mathbf{y}_2 \leq \mathbf{x}_2$, $(\mathbf{x}_1, \mathbf{x}_2) \leq (\mathbf{y}_1, \mathbf{y}_2)$ or $(\mathbf{y}_1, \mathbf{y}_2) \leq (\mathbf{x}_1, \mathbf{x}_2)$. □

6.12. Definition. Let $X_1, X_2, \ldots, X_n$ be partially ordered sets. The *lexicographic order* on $X = \prod_{i=1}^{n} X_i$ is defined as follows. For $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n), \mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$, we say that $\mathbf{x} \leq \mathbf{y}$ if

(1) either $\mathbf{x}_1 < \mathbf{y}_1$ or
(2) $\mathbf{x}_1 = \mathbf{y}_1$ and $\mathbf{x}_2 < \mathbf{y}_2$ or
(3) $\mathbf{x}_1 = \mathbf{y}_1$, $\mathbf{x}_2 = \mathbf{y}_2$ and $\mathbf{x}_3 < \mathbf{y}_3$ or
(...) ... ...
(n) $\mathbf{x}_1 = \mathbf{y}_1, \ldots, \mathbf{x}_{n-1} = \mathbf{y}_{n-1}$ and $\mathbf{x}_n \leq \mathbf{y}_n$.

6.13. Example.

(1) $\mathbb{R}^n$ with lexicographic order is a linearly ordered set.
(2) For $z = r_1 e^{\mathbf{i}\theta_1}$ and $w = r_2 e^{\mathbf{i}\theta_2}$, where $r_1, r_2 \geq 0$ and $0 \leq \theta_1, \theta_2 < 2\pi$, define $z \leq w$ if either $r_1 < r_2$ or $r_1 = r_2$ and $\theta_1 \leq \theta_2$. Then $\mathbb{C}$ is linearly ordered.

(3) The words in the dictionary are arranged in the lexicographic order. That is the reason to use the name *dictionary order*.

6.14. Definition. Let $(X, \leq)$ be a poset and $S \subset X$.

(1) An element $a \in X$ is said to be an *upper bound* of S if $x \leq a$ for every $x \in S$.
(2) An element $a \in X$ is said to be a *lower bound* of S if $a \leq x$ for every $x \in S$.
(3) S is said to be *bounded above* if there is an upper bound of S in X. S is said to be *bounded below* if there is a lower bound of S in X. S is said to be *bounded* if it is bounded above as well as bounded below.

6.15. Definition. Let $(X, \leq)$ be a poset and $S \subset X$.

(1) An upper bound a of S is called the *least upper bound* or *supremum*, denoted by $\sup S$ or $\operatorname{lub} S$, if $a \leq b$ for every upper bound b of S.
(2) A lower bound a of S is called the *greatest lower bound* or *infimum*, denoted by $\inf S$ or $\operatorname{glb} S$, if $b \leq a$ for every lower bound b of S.
(3) If $\sup S$ exists and is in S, it is called the *maximum* or the *largest element* of S and is denoted by $\max S$.
(4) If $\inf S$ exists and is in S, it is called *minimum* or *least element* or *smallest element* of S and is denoted by $\min S$.
(5) If $\max X$ exists, we define $\inf \emptyset = \max X$. If $\max X$ does not exist, we introduce the symbol ∞ and define $\sup X = \inf \emptyset = \infty$.
(6) If $\min X$ exists, we define $\sup \emptyset = \min X$. If $\min X$ does not exist, we introduce the symbol $-\infty$ and define $\inf X = \sup \emptyset = -\infty$.

6.16. Definition. Let $(X, \leq)$ be a poset and $S \subset X$.

(1) An element $a \in S$ is called a *maximal element of S* if $a = b$ whenever $b \in S$ and $a \leq b$.
(2) An element $a \in S$ is called a *minimal element of S* if $a = b$ whenever $b \in S$ and $b \leq a$.

6.17. Examples. $(X, \leq)$ is a partially ordered (sometimes linearly ordered) set. We describe the properties involving elements like maximal, maximum, supremum and upper bound etc. The corresponding versions for minimal, minimum, infimum and lower bound are true.

(1) $\emptyset$ is bounded below as well as bounded above. In fact, every element is a lower bound as well as an upper bound of $\emptyset$. However, if every element of X is an upper bound as well as a lower bound of $A \subset X$, then we cannot conclude that $A = \emptyset$. For example $X = \{2\}$, a singleton with the obvious linear order $2 \leq 2$. Then $A = \{2\}$ is bounded above as well as below with every element (viz. 2) of X being an upper as well as lower bound of A.
(2) If $\emptyset \neq A \subset X$, then every lower bound of A is less than or equal to every upper bound of A. Let a be a lower bound of A and b be an upper bound of A. Since A is not empty, there is $c \in A$. Hence $a \leq c \leq b$ and by transitivity, $a \leq b$. In particular, $\inf A \leq \sup A$, whenever $A \neq \emptyset$ and both $\inf A$ and $\sup A$ exist.

(3) $\sup\emptyset \leq \inf\emptyset$. In $\mathbb{R}$ $\sup\emptyset, \inf\emptyset$ do not exist in $\mathbb{R}$. But still $\sup\emptyset < \inf\emptyset$. In fact, $\inf\mathbb{R} = -\infty = \sup\emptyset < \inf\emptyset = \infty = \sup\mathbb{R}$.
(4) If any of the max, min, sup, inf, exists for a set, then it is unique. We exhibit the proof in one case.
For $A \subset X$, suppose there are at least two $\sup A$, say a and b. Then a is an upper bound of A and b is the least upper bound, and hence, $b \leq a$. By interchanging the rôles of a, b, we get $a \leq b$.
(5) Contrary to the above, maximal and minimal elements of a set, if exist, need not be unique. For example consider $X = \{\{1\}, \{2\}, \{1, 2, 3\}\}$ be the poset with the usual subset relation. In this case, $\{1\}$ and $\{2\}$ are minimal elements of X and $\{1, 2, 3\}$ is a unique maximal as well as maximum element of X.
(6) From the preceding remark, one may tend to conclude that if a maximal element is unique, then it will be the maximum. However it is not the case. For example, consider $\mathbb{N} \cup \{a\}$, where $\mathbb{N}$ has the usual order and for $n \in \mathbb{N}$, declare that $n \leq a$ if and only if $n \leq 100$. Also declare $a \leq a$. Then a is the unique maximal element of $\mathbb{N} \cup \{a\}$. But, $\mathbb{N} \cup \{a\}$ does not have any maximum element.
(7) Maximum of a set is also its maximal element.
(8) $1 = \sup(0, 1)$ but $(0, 1)$ does not have maximum.

6.18. Example. If a finite partially ordered set has a unique maximal element, then it is the maximum.

Indeed let $X = \{a_1, a_2, \ldots, a_n\}$ be a finite partially ordered set with unique maximal element, say a_1. If $a_i \leq a_1$ for $1 \leq i \leq n$, then $a_1 = \max X$. So, suppose to the contrary, that k is the smallest integer such that $a_k \not\leq a_1$. If $a_k \not\leq a_i$ for $i > k$, then a_k is maximal, a contradiction. So, we choose the least $k_1 > k$ such that $a_k < a_{k_1}$. Note that if $a_{k_1} \leq a_1$, then $a_k \leq a_1$, which is not the case. Thus $a_{k_1} \not\leq a_1$. Continuing inductively we find the smallest $k_2 > k_1$, smallest $k_3 > k_2$ and so on such that $a_{k_i} < a_{k_{i+1}}$, $(i \geq 1)$. When this process terminates, we get a maximal element $a_{k_{i+1}}$, a contradiction. Thus a_1 must be the maximum element.

6.19. Example. One of the standard examples of a poset is a powerset of a set ordered by the usual subset order.

Many times a poset is represented by a diagram like the one shown in the Figure 8. The partial order is generated by defining $x_i \leq x_j$ if there is a sequence of line segments joining x_i and x_j, and x_j is above x_i. Then completing the relation by using transitivity and reflexivity.

FIGURE 8. POSET

6.20. Definition. Let $(X, \leq)$ be a poset and $Y \subset X$. If the order $\leq$ on Y is a total order, then $(Y, \leq)$ is called a *chain*.

The following lemma is equivalent to that of axiom of choice.

6.21. Lemma. (Zorn's Lemma) *Let $(X. \leq)$ be a partially ordered set. If every chain in X has an upper bound, then X has a maximal element.*

6.22. Definition. Let $(X, \leq)$ be a poset and $x, y \in X$. If $x < y$, then x is called a *predecessor* of y and y is called a *successor* of x. Further, if there is no element $z \in X$ such that $x < z < y$, then x is called the *immediate predecessor* of y and y is called a *immediate successor* of x.

6.23. Definition. A linearly ordered set X is said to be *well-ordered* if every nonempty subset of X has the least element.

6.24. Example.

(1) The least element of a well-ordered set does not have a predecessor, that is, it is not a successor of any other element.
(2) The set $\{1, 2, 3\}$ with intuitive order, is well-ordered. But 3 does not have a successor, that is, 3 is not a predecessor of any element.
(3) The set $\mathbb{N} = \{1, 3, 5, 7, \ldots, 2, 4, 6, 8, \ldots\}$ in the order of its listing is a well-ordered set. 1 is not a successor of any element, and hence, it does not have a predecessor. Also, 2 has infinitely many predecessors but no immediate predecessor. Consequently, 2 is a successor of infinitely many elements but not the immediate successor of any of them.
(4) $\mathbb{Z} = \{\cdots -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is not a well-ordered set because it does not have the least element.

6.25. Definition. Let $(X, \leq)$ be a totally ordered set. For $a, b \in X$, with $a < b$ we define the following sets.

(1) $(a, b) = \{x \in X : a < x < b\}$, called an *open interval*.
(2) $[a, b] = \{x \in X : a \leq x \leq b\}$, called a *closed interval*.
(3) $[a, b) = \{x \in X : a \leq x < b\}$, called a *closed open interval*.
(4) $(a, b] = \{x \in X : a < x \leq b\}$, called an *open closed interval*.
(5) $(a, \infty) = \{x \in X : a < x\}$, called an *open right ray*.
(6) $(-\infty, a) = \{x \in X : x < a\}$ called an *open left ray*.
(7) $[a, \infty) = \{x \in X : a \leq x\}$, called a *closed right ray*.
(8) $(-\infty, a] = \{x \in X : x \leq a\}$ called a *closed left ray*.
(9) $(-\infty, \infty) = X$.

6.26. Definition. Let X be a well-ordered set with the least element 0. Let $a \in X$. The set $X_a = (-\infty, a) = [0, a)$ is called a *section of X*..

The following is due to Zermelo.

6.27. Theorem. *Every set can be well-ordered.*

The following theorem records the existence of the first uncountable ordinal.

6.28. Theorem. *There is an uncountable well-ordered set X with the largest element ω_1 and smallest element 0 such that X_{ω_1} is uncountable but X_a is countable for every $a \in X \setminus \omega_1$.*

PROOF. Let Y be an uncountable well-ordered set. Consider $Z = \{1, 2\} \times Y$ with the lexicographic order. Let 0 denote the smallest element of Z. Clearly, for any $y \in Y$, and $a = (2, y)$, the section X_a is uncountable. Define

$$A = \{a \in Z : Z_a \text{ is uncountable}\}.$$

Then $\emptyset \neq A \subset Y$. For the smallest element ω_1 of A, $[0, \omega_1]$ is the required set. □

6.29. Remark. The set $[0, \omega_1]$ in the Theorem 6.28 is unique up to order isomorphism.

6.30. Definition. The *first uncountable ordinal* is denoted by ω_1. Also, the set $[0, \omega_1]$ is denoted by Ω and the set $\Omega \setminus \{\omega_1\}$ is denoted by Ω_0.

6.31. Theorem. *Let $A \subset \Omega$ be nonempty and countable. Then $\omega_1 \notin A$ if and only if $\sup A < \omega_1$.*

PROOF. If A is finite, then the result is obvious. So, we assume that A is infinite. Suppose $\omega_1 \notin A = \{a_i : i \in \mathbb{N}\}$. Clearly, $[0, a_i]$ is countable, and hence, $B = \cup_{i=1}^{\infty}[0, a_i]$ is countable. Let γ be the smallest element of $\Omega \setminus B$. But then $b \in B$ if and only if $b < \gamma$. Thus the set of all predecessors of γ is B, a countable set. Hence $\gamma \neq \omega_1$. Also $A \subset B$ and γ is an upper bound of B. Hence $\sup A \leq \sup B \leq \gamma < \omega_1$. The converse is at once. □

6.32. Definition. Let $(I, \leq)$ be a linearly ordered set and $\{X_i : i \in I\}$ be a family of partially ordered sets. For $\mathbf{x}, \mathbf{y} \in \prod_{i\in I} X_i$, define $\mathbf{x} \leq \mathbf{y}$ if either for all $i \in I$, $\mathbf{x}_i \leq \mathbf{y}_i$ or for some $i \in I$, $\mathbf{x}_i < \mathbf{y}_i$ and $\mathbf{x}_j \leq \mathbf{y}_j$ for all $j < i$. $\leq$ is called the *dictionary order* on $\prod_{i\in I} X_i$. In this case, $\prod_{i\in I} X_i$ is called a *linearly ordered product of partially ordered sets*. Also on $\coprod_{i\in I} X_i$ the *dictionary order* is defined for $(x, i), (y, j) \in \coprod_{i\in I} X_i$, $(x, i) \leq (y, j)$ if either $i < j$ or $i = j$ and $x \leq y$ in X_i.

6.33. Proposition. (The permanence properties of order)

(1) Subset of a partially (respectively, linearly, well) ordered set is partially (respectively, linearly, well) ordered.
(2) Linearly ordered product of partially ordered sets is partially ordered with the dictionary order.
(3) Well-ordered product of linearly ordered sets is linearly ordered with the dictionary order. However, linearly ordered product of linearly ordered sets need not be linearly ordered.
(4) Partially ordered coproduct of partially ordered sets is partially ordered with the dictionary order.
(5) Linearly ordered coproduct of linearly ordered sets is linearly ordered with the dictionary order.
(6) Well-ordered coproduct of well-ordered sets is well-ordered with the dictionary order.

PROOF. ☞(1) is a straight forward verification.

☞(2) Let $(I, \leq)$ be a linearly ordered set and $\{X_i : i \in I\}$ be a family of the partially ordered sets.

For $\mathbf{x} = (\mathbf{x}_i) \in \prod_{i\in I} X_i$, clearly $\mathbf{x} \leq \mathbf{x}$ because for each i, $\mathbf{x}_i \leq \mathbf{x}_i$.

Let $\mathbf{x}, \mathbf{y} \in \prod_{i\in I} X_i$ such that $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} \leq \mathbf{x}$. Suppose $\mathbf{x} \neq \mathbf{y}$. Since $\mathbf{x} \leq \mathbf{y}$, and $\mathbf{x} \neq \mathbf{y}$, there is $i \in I$ such that $\mathbf{x}_i < \mathbf{y}_i$ and $\mathbf{x}_j \leq \mathbf{y}_j$ for all $j \leq i$. Since $\mathbf{y} \leq \mathbf{x}$, there is $k \in I$ such that $\mathbf{y}_k \leq \mathbf{x}_k$ and $\mathbf{x}_j \leq \mathbf{y}_j$ for all $j \leq k$. Without loss of generality, we assume that $i < k$. But then $\mathbf{x}_i \leq \mathbf{y}_i$. This contradiction assures that $\mathbf{x} = \mathbf{y}$.

Let $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} \leq \mathbf{z}$. If $\mathbf{x} = \mathbf{y}$ or $\mathbf{y} = \mathbf{z}$, then we are done. So assume that $\mathbf{x} < \mathbf{y}$ and $\mathbf{y} < \mathbf{z}$. Then there are i, k such that $\mathbf{x}_i < \mathbf{y}_i$ and $\mathbf{x}_j \leq \mathbf{y}_j$ for all $j < i$ and $\mathbf{y}_k < \mathbf{z}_k$ and $\mathbf{y}_j \leq \ mz_j$ for all $j < k$. Without loss of generality, assume that $i < k$. Then $\mathbf{x}_i < \mathbf{y}_i \leq \mathbf{z}_i$ and $\mathbf{x}_j \leq \mathbf{y}_j \leq \mathbf{z}_j$ for all $j < i$. So, $\mathbf{x} \leq \mathbf{z}$. Hence $\prod_{i\in I} X_i$ is a partially ordered set.

☞(3) Suppose I is well-ordered and each X_i is linearly ordered. Let $\mathbf{x}, \mathbf{y} \in \prod_{i\in I} X_i$. If $\mathbf{x}_i = \mathbf{y}_i$ for every i, then $\mathbf{x} = \mathbf{y}$. Otherwise, let $I_0 = \{i \in I : \mathbf{x}_i \neq \mathbf{y}_i\}$. Let i_0 be the least element of I_0.[4] By linearity of the order on X_{i_0}, $\mathbf{x}_{i_0} < \mathbf{y}_{i_0}$ or $\mathbf{y}_{i_0} < \mathbf{x}_{i_0}$, so that $\mathbf{x} < \mathbf{y}$ or $\mathbf{y} < \mathbf{x}$.

Let $I = \mathbb{Z}$ and $X_n = \{1, 2\}$ for every $n \in \mathbb{N}$. Define $\mathbf{x}, \mathbf{y}$ in $\prod_{n=-\infty}^{\infty} X_n$ by

$$\mathbf{x}_n = \begin{cases} 1, & \text{if } n \equiv 0 \mod 2; \\ 2, & \text{otherwise;} \end{cases} \quad \textit{and} \quad \mathbf{y}_n = \begin{cases} 2, & \text{if } n \equiv 0 \mod 2; \\ 1, & \text{otherwise.} \end{cases}$$

Clearly, $\mathbf{x} \neq \mathbf{y}$. If $\mathbf{x} < \mathbf{y}$, then there exists $i \in \mathbb{Z}$ such that $\mathbf{x}_i < \mathbf{y}_i$ and $\mathbf{x}_j \leq \mathbf{y}_j$ for all $j < i$. Thus $\mathbf{x}_i$ must be 1 and $\mathbf{y}_j$ must be 2. But then $2 = \mathbf{x}_{i-1} \leq \mathbf{y}_{i-1} = 1$, a contradiction. Thus $\mathbf{x} \not< \mathbf{y}$. Similarly, $\mathbf{y} \not< \mathbf{x}$.

☞(4) Clearly, $\leq$ is reflexive. Let $(x, i) \leq (y, j)$ and $(y, j) \leq (x.i)$. If $i = j$, then the antisymmetry on X_i shows that $(x, i) = (y, j)$. Otherwise $i < j$ and $j < i$ is a contradiction. Now assume that $(x, i) \leq (y, j)$ and $(y, j) \leq (z, k)$. If $i = j = k$,

[4] Essential use of well-ordering of I.

then the transitivity of order on X_i proves the result. Otherwise, $i < k$. Thus we are done.

☞(5) Assume that I is linearly ordered. Clearly, $\coprod_{i\in I} X_i$ is partially ordered by (4). Let $(x,i),(y,j) \in \coprod X_i$. Since I is linearly ordered, we assume without loss of generality that $i \leq j$. If $i = j$, then $x, y \in X_i$, which is linearly ordered. Hence x, y are comparable. So, are (x,i) and (y,i). If $i < j$, then $(x,i) < (y,j)$. Thus $\coprod_{i\in I} X_i$ is linearly ordered.

☞(6) Assume that I and for each $i \in I$, X_i too are well-ordered. Let $A \subset \prod_{i\in I} X_i$ be nonempty. Let i be the least element of I such that $X_i^* \cap A \neq \emptyset$, where $X_i^* = \{(x,i) : x \in X_i\}$. Then the least $x \in X_i$ such that $(x,i) \in A$ gives the least element of A. □

Problems

36. Let $X = \{1,2,3,4,5,6\}$. Find a highest length chain in $\mathscr{P}(X)$.

37. Give an example of a relation R on a set X such that $R \cup \Delta$ is a partial order but R is not a strict order.

38. Give an example of a relation R on a set X such that $R \smallsetminus \Delta$ is a strict order but R is not a partial order.

39. Prove the following.

(1) Every subset of $\mathbb{N}$ is bounded below.
(2) A subset of $\mathbb{N}$ is bounded above if and only if it is finite.
(3) A bounded subset of a poset can contain at most one of its upper bounds.

40. Give examples of

(1) An infinite bounded subset of $\mathbb{R}$.
(2) A bounded subset of $\mathbb{R}$ which does not contain its upper bound.

41. Find the supremum and the infimum of each of these subsets of $\mathbb{R}$ if they exist. $\{\frac{n}{n+1} : n \in \mathbb{N}\}$, $\{x \in \mathbb{Q} : x^2 < 3\}$, $\{0.1, 0.11, 0.111, \dots\}$ and $\{0.1, 0.121, 0.12121, \dots\}$.

42. Prove that a supremum of a set, if exists, is unique.

43. Give an example of a poset in which there are seven maximal elements and three minimal elements.

44. Show that a maximal element of a linearly ordered set is the supremum.

45. Let $A \subset \mathbb{R}$ be bounded above and $a = \sup A$. Given $n \in \mathbb{N}$, show that $a - \frac{1}{n}$ is not an upper bound of A.

46. Let $A \subset \mathbb{R}$ be bounded above and B be the set of all upper bounds of A. Show that $B = [a, \infty)$ for some $a \in \mathbb{R}$. Also, show that $A \cap B \subset \{a\}$.

47. Let the subsets $A = \{a_n : n \in \mathbb{N}\}, B = \{b_n : n \in \mathbb{N}\}$ of $\mathbb{R}$ be bounded above. Show that

$$\sup\{a_n + b_n : n \in \mathbb{N}\} \leq \sup\{a_n : n \in \mathbb{N}\} + \sup\{b_n : n \in \mathbb{N}\}.$$

Also show that the equality may not hold.

48. Let $A, B \subset \mathbb{R}$. If $A \subset B$, is it true that $\sup A \leq \sup B$? Prove your claim.

49. Let $A \subset \mathbb{R}$. Is it true that $\inf A \leq \sup A$? Prove your claim.

50. Let X be a well-ordered set. Give arguments to prove or disprove the following.

(1) For $a, b \in X$ with $a < b$, the interval $(a,b) = \emptyset$ if and only if b is the immediate successor of a.

(2) X is an infinite well-ordered set if and only if every $a \in X$ has a successor.
(3) X is a finite well-ordered set if and only if exactly one element of X does not have any successor.
(4) There exists at most one element of X which does not have any successor.
(5) There could be more than one element of X having no immediate predecessor.

CHAPTER 2

Basic Concepts

7. Topological Spaces

7.1. Definition. Let X be a set and $\mathscr{T}$ be a collection of subsets of X satisfying the following properties.

(1) $\emptyset, X \in \mathscr{T}$.
(2) $\{G_i : i \in I\} \subset \mathscr{T} \Rightarrow \cup_i G_i \in \mathscr{T}$.
(3) $G, H \in \mathscr{T} \Rightarrow G \cap H \in \mathscr{T}$.

Then $\mathscr{T}$ is called a *topology* on X. Also, the pair $(X, \mathscr{T})$ is called a *topological space*. An element of $\mathscr{T}$ is called an *open set* or *$\mathscr{T}$-open set*.

7.2. Remark.

(1) Now onwards, unless explicit mentioned about possibility of emptiness of a set, it is assumed that a set considered is a nonempty set.
(2) Usually we write "X is a topological space" instead of "$(X, \mathscr{T})$ is a topological space". In this case, it is assumed that the underlying topology on X is $\mathscr{T}$.

7.3. Example ([1]). Let $X = \{a, b, c\}$ be a set of three elements. The following are some of the topologies defined on the set X which are pictorially demonstrated below.

(1) $\mathscr{T}_1 = \{\emptyset, X\}$ (Figure 9(a)).
(2) $\mathscr{T}_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$ (Figure 9(b)).
(3) $\mathscr{T}_3 = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, X\}$ (Figure 9(c)).
(4) $\mathscr{T}_4 = \{\emptyset, \{b\}, X\}$ (Figure 9(d)).
(5) $\mathscr{T}_5 = \{\emptyset, \{a\}, \{b, c\}, X\}$ (Figure 9(e)).
(6) $\mathscr{T}_6 = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$ (Figure 9(f)).
(7) $\mathscr{T}_7 = \{\emptyset, \{a, b\}, X\}$ (Figure 9(g)).
(8) $\mathscr{T}_8 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ (Figure 9(h)).
(9) $\mathscr{T}_9 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$ (Figure 9(i)).

[1] Courtesy Jay

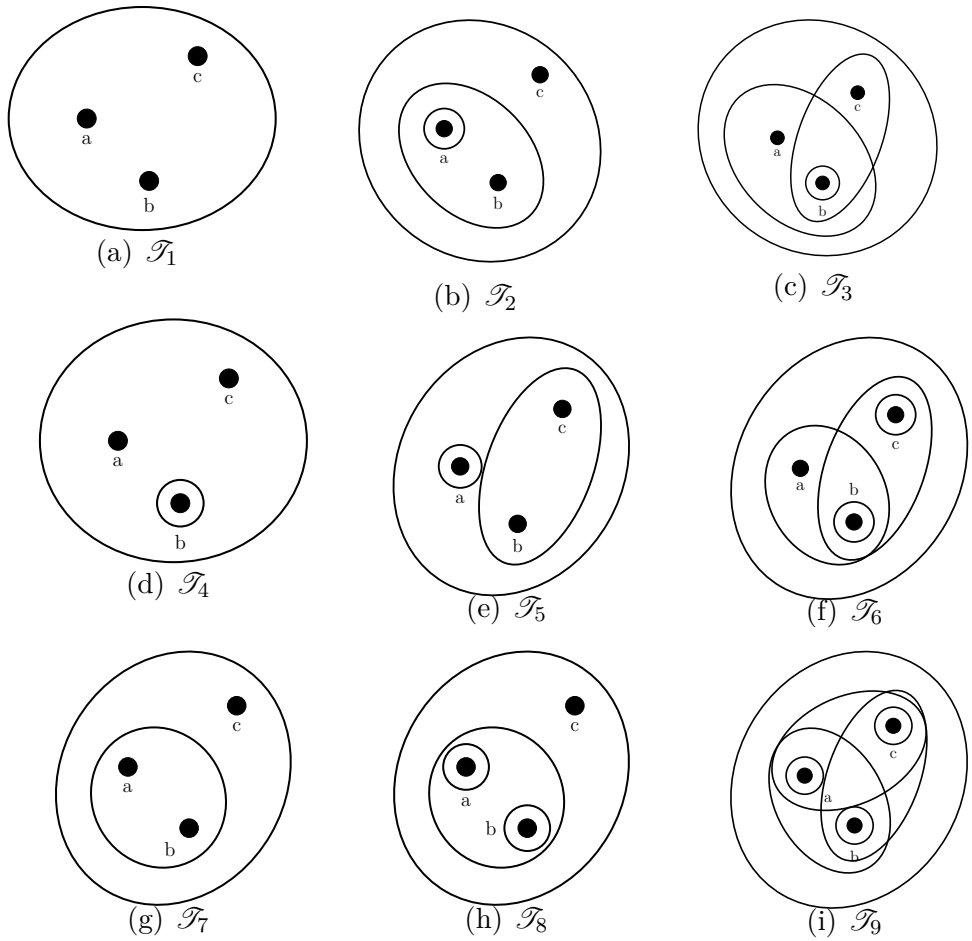


FIGURE 9. Some Topologies on the Set $X = \{a, b, c\}$

The following Lemma shows that something stronger than condition (3) of the Definition 7.1 is true.

7.4. Lemma. *Let $(X, \mathscr{T})$ be a topological space. Then $\mathscr{T}$ is closed under finite intersection.*

PROOF. Let $G_1, G_2, \ldots, G_n \in \mathscr{T}$. Applying (3) to G_1, G_2, we get $G_1 \cap G_2 \in \mathscr{T}$. Now applying the Definition 7.1 (3) to $G_1 \cap G_2$ and G_3. we get $(G_1 \cap G_2) \cap G_3 \in \mathscr{T}$. Continuing in this way, in finitely many steps, we get, $G_1 \cap G_2 \cap \cdots \cap G_n \in \mathscr{T}$. This completes the proof. □

7.5. Examples. Let X be a set. Then

(1) $\mathscr{I} = \{\emptyset, X\}$ is a topology on X.
(2) $\mathscr{D} = \mathscr{P}(X)$, the powerset of X, is a topology on X.

7.6. Definition. Let X be a set. The topology $\mathscr{I}$ defined in the Example 7.5(1) is called the *indiscrete topology* or *trivial topology* on X and the topology $\mathscr{D}$ defined in the Example 7.5(2) is called the *discrete topology* on X.

7.7. Example. Let $X = \{a, b\}$, a two point set. Then $\mathscr{T} = \{\emptyset, \{a\}, \{a, b\}\}$ is a topology on $\{a, b\}$. This topological space is called the *Sierspińsky space*. The Sierspińsky topology can also be described in the following ways.

$$\mathscr{T} = \{G \subset X : G = \emptyset \text{ or } a \in G\}. \tag{7.7.1}$$

$$= \{G \subset X : G = X \text{ or } b \notin G\}. \tag{7.7.2}$$

In general, on a set X and a point $a \in X$, (7.7.1) defines a topology, called a *particular point topology* or a *VIP topology*, where VIP is an acronym for *Very Important Point*. Also for a point $b \in X$, (7.7.2) defines a topology, called an *excluded point topology*.

7.8. Definition. Let $\mathscr{T}_1, \mathscr{T}_2$ be two topologies on a set X. If $\mathscr{T}_1 \subset \mathscr{T}_2$, then we say that $\mathscr{T}_1$ is *weaker* or *coarser* than $\mathscr{T}_2$ and write $\mathscr{T}_1 \leq \mathscr{T}_2$. In this case, we also say that $\mathscr{T}_2$ is *stronger* or *finer* than $\mathscr{T}_1$. If $\mathscr{T}_1 \subsetneq \mathscr{T}_2$, then we say that $\mathscr{T}_1$ is *strictly weaker or strictly coarser* than $\mathscr{T}_2$ and write $\mathscr{T}_1 < \mathscr{T}_2$. In this case, we also say that $\mathscr{T}_2$ is *strictly stronger* or *strictly finer* than $\mathscr{T}_1$. We say that $\mathscr{T}_1$ and $\mathscr{T}_2$ are *comparable* if $\mathscr{T}_1 \leq \mathscr{T}_2$ or $\mathscr{T}_2 \leq \mathscr{T}_1$.

7.9. Example. Let X be a set. Define

$$\mathscr{T}_f = \{G \subset X : G = \emptyset \text{ or } X \setminus G \text{ is finite}\}. \tag{7.9.1}$$

Then $\mathscr{T}_f$ is a topology on X.

From (7.9.1), it is clear that $\emptyset \in \mathscr{T}_f$. Also $X \setminus X = \emptyset$, a finite set. So, $X \in \mathscr{T}_f$. Now let $\{G_i : i \in I\}$ be a collection of members of $\mathscr{T}_f$ and $G = \cup\{G_i : i \in I\}$. If $G = \emptyset$, then we are done. Otherwise, $G_j \neq \emptyset$ for some j. But then $X \setminus G_j$ is finite. Also, $X \setminus G \subset X \setminus G_j$. Hence $X \setminus G$ is finite. So, $G \in \mathscr{T}_f$.

Let $G, H \in \mathscr{T}_f$. If $G \cap H = \emptyset$, then we are done. Otherwise $X \setminus G, X \setminus H$ are finite. But then $X \setminus (G \cap H) = (X \setminus G) \cup (X \setminus H)$, which is finite. Thus $G \cap H \in \mathscr{T}_f$.

7.10. Definition. The topology $\mathscr{T}_f$ considered in (7.9.1) is called the *finite complement topology* or *cofinite topology* and $(X, \mathscr{T}_f)$ is called the *cofinite topological space*.

7.11. Example. The cofinite topology on a finite set X is discrete.

Let $G \subset X$. Then $X \setminus G$ is finite. Hence $G \in \mathscr{T}_f$. Thus $\mathscr{D} \leq \mathscr{T}_f$.

7.12. Example. Let X be a set. Similar to the Example 7.9,

$$\mathscr{T}_c = \{G \subset X : G = \emptyset \text{ or } X \setminus G \text{ is countable}\} \tag{7.12.1}$$

is a topology on X.

7.13. Definition. The topology $\mathscr{T}_c$ considered in (7.12.1) is called the *countable complement topology* or *cocountable topology* and $(X, \mathscr{T}_c)$ is called the *cocountable topological space*.

More generally, for each infinite cardinal number α, we can define the following two topologies.

7.14. Example. Let X be a set and α be an infinite cardinal. Define

$$\mathscr{T}_{<\alpha} = \{G \subset X : G = \emptyset \text{ or } |X \setminus G| < \alpha\}$$

and

$$\mathscr{T}_{\alpha} = \{G \subset X : G = \emptyset \text{ or } |X \setminus G| \leq \alpha\}.$$

Then $\mathscr{T}_{<\alpha}$ and $\mathscr{T}_{\alpha}$ are topologies on X.

7.15. Definition. We call the topology $\mathscr{T}_{\alpha}$ the *co α-topology* and the topology $\mathscr{T}_{<\alpha}$ the *cosub α-topology*. Extending the same terminology X equipped with $\mathscr{T}_{\alpha}$ is called a *co α-topological space* and X equipped with $\mathscr{T}_{<\alpha}$ is called a *cosub α-topological space*

7.16. Remark.

(1) $\mathscr{T}_f = \mathscr{T}_{<\aleph_0} \leq \mathscr{T}_{\aleph_0} = \mathscr{T}_c$.
(2) The topologies of the form either $\mathscr{T}_{<\alpha}$ or $\mathscr{T}_{\alpha}$ for some infinite cardinal number α are called the *cardinal topologies*.
(3) One can easily see that if α and β are two infinite cardinals such that $\alpha < \beta$, then
$$\mathscr{T}_{<\alpha} \leq \mathscr{T}_{\alpha} \leq \mathscr{T}_{<\beta} \leq \mathscr{T}_{\beta}.$$
(4) $|X| < \alpha$ if and only if $\mathscr{T}_{<\alpha} = \mathscr{T}_{\alpha} = \mathscr{D}$.

7.17. Example. Let X be a set. Choose $A \subset X$. Then $\mathscr{T} = \{\emptyset, X, A\}$ is a topology on X. $\mathscr{T} = \mathscr{I}$ if and only if $A = \emptyset$ or $A = X$. On the other hand $\mathscr{I} = \mathscr{T} = \mathscr{D}$ if and only if X is singleton.

7.18. Example. Let X be a nonempty set and $\{A_i : 1 \leq i \leq n\}$ be a chain of subsets of X. Then $\mathscr{T} = \{\emptyset, X, A_1, A_2, \ldots, A_n\}$ is a topology on X. Also $\mathscr{I} \leq \mathscr{T} \leq \mathscr{D}$.

7.19. Remark. If we replace a finite chain by an infinite chain in the Example 7.18, then $\mathscr{T}$ need not be a topology because an infinite chain need not be closed under union. For example the chain $\{(7 + \frac{1}{n}, 10 - \frac{1}{n}) : n \in \mathbb{N}\}$ is not closed under union.

7.20. Example. For a set X and $\emptyset \subsetneqq A \subsetneqq X$, $\mathscr{T} = \{\emptyset, X, A, X \setminus A\}$ is a topology on X.

7.21. Example. For a set X and $A \subset X$, $\mathscr{T} = \{G \subset X : G = \emptyset \text{ or } A \subset G\}$ is a topology on X.

7.22. Example. For a set X, $A \subset X$, $\mathscr{T} = \{G \subset X : G = X \text{ or } G \subset A\}$ is a topology on X.

Problems

51. Which one of the following is a topology on a given set X?

(1) $\mathscr{T} = \{G \subset X : G \text{ is finite}\}$.
(2) $\mathscr{T} = \{G \subset X : G \text{ is infinite}\}$.

52. Let X be a set and $a \in X$. Consider the VIP topology

$$\mathscr{T} = \{U \subset X : U = \emptyset \text{ or } a \in U\}$$

on X defined in the Example 7.7.

(1) Show that $\mathscr{T}$ is a topology on X.
(2) Show that $\mathscr{T}$ is closed under arbitrary intersection.

53. Let $\mathscr{T} = \{G \subset \mathbb{R} : G = \emptyset \text{ or } G = \mathbb{R} \text{ or } G = (-r, r) \text{ for some } r > 0\}$. Show that $\mathscr{T}$ is a topology on $\mathbb{R}$.

54. Let $\mathscr{T} = \{G \subset \mathbb{R} : G = \emptyset \text{ or } G = \mathbb{R} \text{ or } G = (-n, n) \text{ for some } n \in \mathbb{N}\}$. Show that $\mathscr{T}$ is a topology on $\mathbb{R}$.

55. Let $\mathscr{T} = \{G \subset \mathbb{R} : G = \emptyset \text{ or } G = \mathbb{R} \text{ or } G = (-r, r) \text{ for some } r \in \mathbb{Q}, r > 0\}$. Show that $\mathscr{T}$ is not a topology on $\mathbb{R}$.

56. Let $\mathscr{T} = \{G \subset \mathbb{R} : G = \emptyset \text{ or } G = \mathbb{R} \text{ or } G = (r, \infty) \text{ for some } r \in \mathbb{R}\}$. Show that $\mathscr{T}$ is a topology on $\mathbb{R}$.

57. Let $\mathscr{T} = \{G \subset \mathbb{R} : G = \emptyset \text{ or } G = \mathbb{R} \text{ or } G = (r, \infty) \text{ for some } r \in \mathbb{Q}\}$. Show that $\mathscr{T}$ is not a topology on $\mathbb{R}$.

58. List all the topologies on $X = \{a, b, c\}$ containing $\{a\}$.

59. List all the topologies on a set X in which every singleton is open.

60. Show that $\mathscr{T} = \{G \subset \mathbb{R} : G = \emptyset \text{ or } \mathbb{Q} \subset G\}$ is a topology on $\mathbb{R}$.

61. Show that $\mathscr{T} = \{G \subset \mathbb{R} : G = \mathbb{R} \text{ or } G \cap \mathbb{Q} = \emptyset\}$ is a topology on $\mathbb{R}$.

62. Which of $\{\frac{n}{n+1} : n \in \mathbb{N}\}$, $[0, 1]$, $\mathbb{Q}$, $\mathbb{R} \setminus Q$, $\mathbb{R} \setminus \{0\}$ are $\mathscr{T}_f$-open or $\mathscr{T}_c$-open in $\mathbb{R}$?

63. Consider the topology $\mathscr{T}$ on X given in Example 7.18. Obtain the necessary and sufficient condition for each of $\mathscr{I} < \mathscr{T}$ and $\mathscr{T} < \mathscr{D}$.

64. Show that in a cofinite topological space arbitrary intersection of open sets need not be open.

65. Give an example in the following topological spaces to show that arbitrary intersection of open sets need not be open.

(1) $\mathbb{R}$ with the cocountable topology.
(2) $\mathbb{R}$ with the usual topology.
(3) $\mathbb{R}$ with the lower limit topology.

66. Let X be a set and Y be a topological space and $f : X \to Y$ be a function. Show that

$$\mathscr{T} = \{f^{-1}(U) : U \text{ be open in } Y\}$$

is a topology on X.

67. Let X be a set and $\{\mathscr{T}_\alpha : \alpha \in \Lambda\}$ be a family of topologies on X. Show that $\cap\{\mathscr{T}_\alpha : \alpha \in \Lambda\}$ is a topology on X.

68. Let $\mathscr{T}_1, \mathscr{T}_2$ be two topologies on a set X. Is $\mathscr{T}_1 \cup \mathscr{T}_2$ a topology on X? Justify.

69. Let $\{\mathscr{T}_\alpha \mid \alpha \in \Lambda\}$ be a family of topologies on a set X. Show that there is a unique smallest topology on X containing $\mathscr{T}_\alpha$ for all α. Also show that there is a unique largest topology on X contained in $\mathscr{T}_\alpha$ for all α.

70. Prove that $\mathscr{I}$ and $\mathscr{D}$ are topologies on X. Also, prove that $\mathscr{I} = \mathscr{D}$ if and only if X is singleton.

71. Show that in a cardinal topology, a set is open if one of its nonempty subset is open.

72. Let X be a finite set. Show that the cofinite topology coincides with the discrete topology. Also prove the converse.

73. Let X be a countable set. Show that the cocountable topology coincides with the discrete topology. Also prove the converse.

74. Let X be a nonempty set. On X, show that

(1) $\mathscr{T}_f \leq \mathscr{T}_c$.
(2) $\mathscr{T}_c \leq \mathscr{T}_f$ if and only if X is finite.

8. Base

In general, there may be too many open sets in a topological space to handle with. So, we develop a concept of a base for a topology that turns out to be a subcollection of a topology. Instead of dealing with all open sets, it will be sufficient to deal with this subcollection of basic open sets only. One can think of a base as a collection of building blocks.

8.1. Definition. Let $(X, \mathscr{T})$ be a topological space. A collection $\mathscr{B} \subset \mathscr{T}$ is said to be an *open base* for the topology $\mathscr{T}$ if

$$\mathscr{T} = \{\cup\mathscr{C} : \mathscr{C} \subset \mathscr{B}\}.$$

That is, $G \in \mathscr{T}$ if and only if G is a union of members of $\mathscr{B}$. For brevity we shall use the term *base* for an open base. In context of $\mathscr{B}$, an element of $\mathscr{B}$ is called a *basic open set*. $\mathscr{T}$ is called the *topology generated by* $\mathscr{B}$. Some authors use the term *basis* or an *open basis* instead of base.

8.2. Proposition. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{B}$ be a base for $\mathscr{T}$. Then $\mathscr{T}$ is the weakest topology on X such that $\mathscr{B} \subset \mathscr{T}$.*

Proof. Clearly, $\mathscr{B} \subset \mathscr{T}$. Let $\mathscr{T}'$ be a topology on X such that $\mathscr{B} \subset \mathscr{T}'$. Let $G \in \mathscr{T}$. But then $G = \cup B_i$ with $B_i \in \mathscr{B} \subset \mathscr{T}'$. Hence $G = \cup_i B_i \in \mathscr{T}'$. Thus $\mathscr{T} \leq \mathscr{T}'$. □

8.3. Proposition. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{B}$ be a collection of subsets of X. The following are equivalent.*

(1) $\mathscr{B}$ is a base for $\mathscr{T}$.
(2) $\mathscr{T} = \{G \subset X$: for every $x \in G$, there is $B_x \in \mathscr{B}$ such that $x \in B_x \subset G\}$.

Proof. ☞(1) ⇒ (2) Let $x \in G \in \mathscr{T}$. By the definition of base, $G = \cup_i B_i$ with $B_i \in \mathscr{B}$. But then there is some $j \in I$ such that $x \in B_j$. Defining $B_x = B_j$, we are done.

Now, let $G \subset X$ such that for every $x \in G$ there is $B_x \in \mathscr{B}$ such that $x \in B_x \subset G$. Clearly, $\cup B_x = G$, and hence, $G \in \mathscr{T}$.

☞(2) ⇒ (1) Let $B \in \mathscr{B}$. Then for every $x \in B$, we have $x \in B \subset B$. So let $B_x = B$. Thus for every $x \in B$, $x \in B_x \subset B$. So $B \in \mathscr{T}$. That is, $\mathscr{B} \subset \mathscr{T}$.

Now let $G \in \mathscr{T}$. Then for every $x \in G$, there is $B_x \in \mathscr{B}$ such that $x \in B_x \subset G$. Taking union of the sets in the middle over all $x \in G$, we get $\cup B_x = G$. Thus $\mathscr{T} \subset \{\cup\mathscr{C} : \mathscr{C} \subset \mathscr{B}\}$. On the other hand, let $G = \cup_{i\in I} B_i$, where $B_i \in \mathscr{B}$ for every $i \in I$. Let $x \in G$. Then there is $j \in J$ such that $x \in B_j$. Hence $x \in B_j \subset \cup_{i\in I} B_i = G$. So, $G \in \mathscr{T}$. This ensures that $\{\cup\mathscr{C} : \mathscr{C} \subset \mathscr{B}\} \subset \mathscr{T}$.

Summing up all the above, we have a topology $\mathscr{T}$ and $\mathscr{B} \subset \mathscr{T}$ such that $\mathscr{T} = \{\cup\mathscr{C} : \mathscr{C} \subset \mathscr{B}\}$. So, $\mathscr{B}$ is a base for the topology $\mathscr{T}$. □

8.4. Proposition. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{B}$ be a base for $\mathscr{T}$. Then*

(1) $\cup\mathscr{B} = X$.
(2) For $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, there is $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Proof. Indeed, $X \in \mathscr{T}$. So, $X = \cup_i B_i$ with $B_i \in \mathscr{B}$. This proves (1). Now let $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$. Since $\mathscr{B} \subset \mathscr{T}$, $B_1, B_2 \in \mathscr{T}$. Consequently, $B_1 \cap B_2 \in \mathscr{T}$. Hence by the Proposition 8.3, there is $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$. This completes the proof. □

The important part is the converse to the above proposition. It will help us define the topology through bases.

8.5. Proposition. *Let X be a set and $\mathscr{B} \subset \mathscr{P}(X)$ satisfy the following.*

(1) $\cup\mathscr{B} = X$.
(2) For $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$, there is $B_3 \in \mathscr{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

Define

$$\mathscr{T} = \{\cup\mathscr{C} : \mathscr{C} \subset \mathscr{B}\}. \tag{8.5.1}$$

Then $\mathscr{T}$ is a topology and $\mathscr{B}$ is a base for $\mathscr{T}$.

Proof. Note that $\emptyset$ is a union of empty family of members of $\mathscr{B}$ and X is, by hypothesis, a union of members of $\mathscr{B}$. Thus $\emptyset, X \in \mathscr{T}$.

Let $\{G_i : i \in I\} \subset \mathscr{T}$. Since each G_i is a union of members of $\mathscr{B}$, $\cup_i G_i$ is a union of unions of members of $\mathscr{B}$, and hence, union of members of $\mathscr{B}$. Thus $\cup_i G_i \in \mathscr{T}$.

Now let $G, H \in \mathscr{T}$. Let $x \in G \cap H$. Since G is a union of members of $\mathscr{B}$, there is $B_1 \in \mathscr{B}$ such that $x \in B_1 \subset G$. Similarly, there is $B_2 \in \mathscr{B}$ such that $x \in B_2 \subset H$. So, by the hypothesis, there is $B_x \in \mathscr{B}$ such that $x \in B_x \subset B_1 \cap B_2 \subset G \cap H$. Taking union of all such B_x over $x \in G \cap H$, we conclude that $\cup_{x\in G\cap H} B_x = G \cap H$. Thus $G \cap H \in \mathscr{T}$. □

8.6. Corollary. *Let X be a set and $\mathscr{B} \subset \mathscr{P}(X)$ satisfy the following.*
(1) $\cup\mathscr{B} = X$.
(2) $B_1, B_2 \in \mathscr{B} \Rightarrow$ either $B_1 \cap B_2 = \emptyset$ or $B_1 \cap B_2 \in \mathscr{B}$.
Then $\mathscr{B}$ is a base for the topology $\mathscr{T} = \{\cup\mathscr{C} : \mathscr{C} \subset \mathscr{B}\}$.

8.7. Corollary. *Let $(X, \mathscr{T})$ be a topological space. Then $\mathscr{T}$ is a base for $\mathscr{T}$.*

8.8. Proposition. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{B}$ be a base for $\mathscr{T}$. Then $\mathscr{T}$ is the smallest topology on X containing $\mathscr{B}$.*

Proof. Let $\mathscr{T}'$ be a topology on X such that $\mathscr{B} \subset \mathscr{T}'$. Let $G \in \mathscr{T}$. So, G is a union of members of $\mathscr{B}$. Since every $B \in \mathscr{B}$ is $\mathscr{T}'$-open, G is a union of $\mathscr{T}'$-open subsets of X. Hence $G \in \mathscr{T}'$. Hence $\mathscr{T} \subset \mathscr{T}'$. Thus $\mathscr{T}$ is the smallest topology on X containing $\mathscr{B}$. □

8.9. Example. $\mathscr{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$ is a base for a topology on $\mathbb{R}$.

Indeed $\mathbb{R}$ is a union of open intervals. Now consider two open intervals $(a, b), (c, d)$. Let $p = \max\{a, c\}$ and $q = \min\{b, d\}$. Then clearly, $a \leq p$ and $q \leq b$. So $(p, q) \subset (a, b)$. Similarly, $(p, q) \subset (c, d)$. Hence $(p, q) \subset (a, b) \cap (c, d)$. On the other hand, for any $x \in (a, b) \cap (c, d)$, we have $a < x$ and $c < x \Rightarrow p < x$. Also,

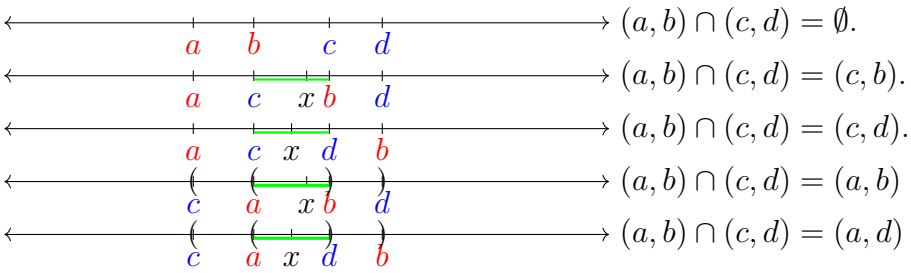


Figure 10. Intersection of Two Intervals

$x < b$ and $x < d \Rightarrow x < q$. This gives $(a, b) \cap (c, d) \subset (p, q)$. That is, in particular, $x \in (p, q) = (a, b) \cap (c, d)$, (see the Figure 10). Thus $(a, b) \cap (c, d) \in \mathscr{B}$. Hence $\mathscr{B}$ is a base for a topology on $\mathbb{R}$.

8.10. Definition. The topology generated by

$$\mathscr{B} = \{(a, b) : a, b \in \mathbb{R}, a < b\}$$

discussed in the Example 8.9 is called the *usual topology*[2] on $\mathbb{R}$.

Important Note

Whenever $\mathbb{R}$ is mentioned without any specific reference to the topology it carries, it is assumed that it carries the usual topology.

[2]Some authors also call it the *standard* topology.

Using the same argument that intersection of two intervals is again an interval a number of different topologies can be constructed. Here the additional requirement is that intersection of two same type of intervals is again the same type of interval. This is more important fact used in the construction of all such topological spaces.

The follows can be proved by mimicking the proof of the Example 8.9.

8.11. Example. $\mathscr{B}_\ell = \{[a, b) : a, b \in \mathbb{R}, a < b\}$ is a base for a topology on $\mathbb{R}$.

8.12. Definition. The topology generated by the base

$$\mathscr{B}_\ell = \{[a, b) : a, b \in \mathbb{R}, a < b\},$$

discussed in the Example 8.11, is called the *lower limit topology* on $\mathbb{R}$ and is denoted by $\mathscr{T}_\ell$. Also, $\mathbb{R}_\ell$ denotes $\mathbb{R}$ with the lower limit topology on it, known as the *Sorgenfrey line*.

The following can be proved by mimicking the proof of the Example 8.9.

8.13. Example. $\mathscr{B}_u = \{(a, b] : a, b \in \mathbb{R}, a < b\}$ is a base for a topology on $\mathbb{R}$.

8.14. Definition. The topology generated by the base

$$\mathscr{B}_u = \{(a, b] : a, b \in \mathbb{R}, a < b\},$$

discussed in the Example 8.13, is called the *upper limit topology* on $\mathbb{R}$ and is denoted by $\mathscr{T}_u$. Also, $\mathbb{R}_u$ denotes $\mathbb{R}$ with the upper limit topology on it.

8.15. Example. For $\mathbf{x} \in \mathbb{R}^2$ and $\varepsilon > 0$, define

$$B(\mathbf{x}, \varepsilon) = \{\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2) \in \mathbb{R}^2 : (\mathbf{y}_1 - \mathbf{x}_1)^2 + (\mathbf{y}_2 - \mathbf{x}_2)^2 < \varepsilon^2\} \tag{8.15.1}$$

and let

$$\mathscr{B} = \{B(\mathbf{x}, \varepsilon) \subset \mathbb{R}^2 : \mathbf{x} \in \mathbb{R}^2, \varepsilon > 0\}. \tag{8.15.2}$$

Then $\mathscr{B}$ is a base for some topology on $\mathbb{R}^2$.

Note that $B(\mathbf{x}, \varepsilon)$ is an open disc of radius ε centred at $\mathbf{x}$. Clearly, for every $\mathbf{x} \in \mathbb{R}^2$, we have $\mathbf{x} \in B(\mathbf{x}, 1) \subset \cup\mathscr{B}$. Thus $\mathbb{R}^2$ is union of members of $\mathscr{B}$.

Also, let $B_1 = B(\mathbf{x}, \varepsilon_1), B_2 = B(\mathbf{y}, \varepsilon_2) \in \mathscr{B}$ and $\mathbf{z} \in B_1 \cap B_2$. Define $\delta_1 = \sqrt{(\mathbf{x}_1 - \mathbf{z}_1)^2 + (\mathbf{x}_2 - \mathbf{z}_2)^2}$, $\delta_2 = \sqrt{(\mathbf{y}_1 - \mathbf{z}_1)^2 + (\mathbf{y}_2 - \mathbf{z}_2)^2}$ and $\delta = \min\{\delta_1, \delta_2\}$.

Then using an easy school mathematics, one shows that $z \in B(\mathbf{z}, \delta) \subset B_1 \cap B_2$, proving that $\mathscr{B}$ is a base for a topology on $\mathbb{R}^2$.

8.16. Definition. The topology generated on $\mathbb{R}^2$ by the base in (8.15.2), discussed in the Example 8.15, is called the *usual topology* or *Euclidean topology* on $\mathbb{R}^2$.

8.17. Example. Let $\mathscr{B} = \{(a, b) \times (c, d) \subset \mathbb{R}^2 : a, b, c, d \in \mathbb{R}, a < b \text{ and } c < d\}$. Then $\mathscr{B}$ is a base for some topology on $\mathbb{R}^2$.

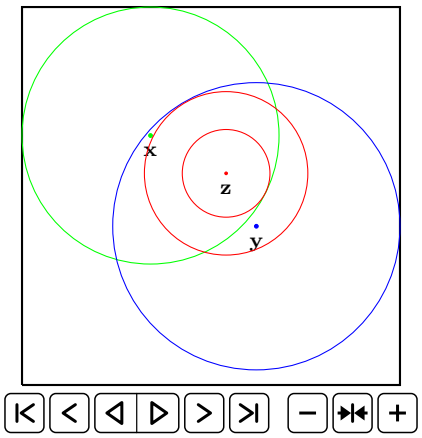


FIGURE 11. Intersection of Two Open Spheres

Indeed, for $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^2$, $(\mathbf{x}_1, \mathbf{x}_2) \in (\mathbf{x}_1 - 1, \mathbf{x}_1 + 1) \times (\mathbf{x}_2 - 1, \mathbf{x}_2 + 1) \in \mathscr{B}$. Hence $\cup\mathscr{B} = \mathbb{R}^2$. Also, for $(a, b)\times(c, d), (e, f)\times(g, h) \in \mathscr{B}$, either their intersection is empty or

$$[(a, b)\times(c, d)]\cap[(e, f)\times(g, h)] = [(a, b)\cap(e, f)]\times[(c, d)\cap(g, h)] = (p, q)\times(r, s),$$

where $p = \max\{a, e\}, q = \min\{b, f\}, r = \max\{c, g\}$ and $s = \min\{d, h\}$. Thus $\mathscr{B}$ is a base for some topology. We shall see in due course of time that this is in fact, the usual topology of $\mathbb{R}^2$.

8.18. Example. Let X be a set and $\mathscr{B}$ be the collection of all singletons of X. Then $\mathscr{B}$ is a base for the discrete topology on X.

Here It is clear at once that $\mathscr{B} \subset \mathscr{D}$ and every $G \in \mathscr{D}$, is a union of singletons, that is, a union of members of $\mathscr{B}$. Thus $\mathscr{B}$ generates the discrete topology.

The following theorem gives an open base for a topology on the upper half plane. The topology is due to Moore.

8.19. Theorem. *Let $X = \mathbb{R} \times [0, \infty) = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. For $(x_0, y_0) \in X$ with $y_0 > 0$ and $0 < r < y_0$ define*

$$B_M((x_0, y_0), r) = \{(x, y) : (x - x_0)^2 + (y - y_0)^2 < r^2\}.$$

Also, for $(x_0, 0)$ and $r > 0$ let

$$B_M((x_0, 0), r) = \{(x, y) : (x - x_0)^2 + (y - r)^2 < r^2\} \cup \{x_0, 0\}.$$

Then

$$\mathscr{B}_M = \{B_M((x, y), r) : \text{ either } y > r > 0 \text{ or } y = 0, r > 0\}.$$

is a base for a topology on X.

PROOF. We appeal the reader to create the proof from the Figure 12. □

8.20. Definition. The above topology, defined in the Theorem 8.19, is called the *Moore topology*. The upper half plane with this topology is called the *Moore plane*.

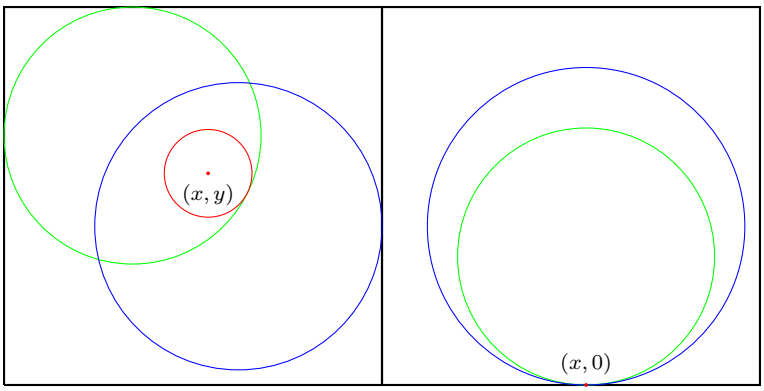


FIGURE 12. Moore Basic Open Sets

8.21. Lemma. *Let $\mathscr{T}, \mathscr{T}'$ be two topologies on a set X generated by the bases $\mathscr{B}, \mathscr{B}'$ respectively. Then the following are equivalent.*

(1) $\mathscr{T}'$ is finer than $\mathscr{T}$.
(2) For every $B \in \mathscr{B}$ and for every $x \in B$, there exists $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.

PROOF. Assume (1). Then $x \in B \in \mathscr{B} \Rightarrow B \in \mathscr{T} \subset \mathscr{T}' \Rightarrow \exists\ B' \in \mathscr{B}'$ with $x \in B' \subset B$. Conversely, assume (2). Then

$$\begin{aligned} G \in \mathscr{T} \text{ and } x \in G &\Rightarrow \exists\ B \in \mathscr{B} \text{ such that } x \in B \subset G \\ &\Rightarrow \exists\ B' \in \mathscr{B}' \text{ such that } x \in B' \subset B \subset G. \\ &\Rightarrow G \in \mathscr{T}' \end{aligned}$$

Thus $\mathscr{T} \subset \mathscr{T}'$. This competes the proof. □

8.22. Corollary. *Let $\mathscr{T}, \mathscr{T}'$ be two topologies on a set X generated by the bases $\mathscr{B}, \mathscr{B}'$ respectively. If $\mathscr{B} \subset \mathscr{B}'$, then $\mathscr{T} \subset \mathscr{T}'$.*

PROOF. Clearly, $\mathscr{B} \subset \mathscr{B}' \subset \mathscr{T}'$. But $\mathscr{T}$ is the smallest topology containing $\mathscr{B}$. Hence $\mathscr{T} \subset \mathscr{T}'$. □

The following topology was introduced by Furstenberg in 1955 [**Fur55**] in one of the shortest articles. Later on various properties of this space were investigated by Lovas and Mező in 2015 in [**LM15**].

8.23. Example. For $a \in \mathbb{Z} \setminus \{0\}, b \in \mathbb{Z}$, define $B_{a,b} = \{na + b : n \in \mathbb{Z}\} = a\mathbb{Z} + b$. Then

$$\mathscr{B}_{ap} = \{B_{a,b} : a, b + 1 \in \mathbb{N}, 0 \leq b < a\} \tag{8.23.1}$$

is a base for some topology.

First we note that $B_{0,b} = \{b\}$. Hence If we allow $a = 0$. Then the $\mathscr{B}_{ap}$ contains all singletons and hence it is a base for the discrete topology. Thus to avoid the discrete topology, we should put the condition $a \in \mathbb{Z} \setminus \{0\}$. Also, It is clear that $B_{a,b} = B_{-a,b}$. So, we can consider, without loss of generality, the sets

$B_{a,b}$, where $a > 0$. Once $a > 0$ is fixed, the sets $B_{a,b}$ are given by $0 \leq b < a$. So, we assume in the definition of $\mathscr{B}_{ap}$ that $a, b+1 \in \mathbb{N}$ and $0 \leq b < a$.

Since $B_{1,0} = \mathbb{Z}$, $\cup\mathscr{B}_{ap} = \mathbb{Z}$. Now let $B_{a,b}, B_{c,d} \in \mathscr{B}_{ap}$ and $m \in B_{a,b} \cap B_{c,d}$.

Thus $m = ak + b$. This gives $m \cong b \mod a$. Similarly, $m \cong d \mod c$. That is there is a solution to the system $x \cong b \mod a$ and $x \cong d \mod c$. Hence by the Chinese remainder theorem, there are infinitely many solutions to this system and all solution are in arithmetic progression with common difference e, where $e = \text{lcm}\{a, c\}$. Thus there is $0 \leq f < e$ such that $m \in B_{e,f} \subset B_{a,b} \cap B_{c,d}$. Thus $\mathscr{B}_{ap}$ is a base for some topology on $\mathbb{Z}$.

8.24. Definition. The topology generated by the base $\mathscr{B}_{ap}$ defined in (8.23.1) is called the *arithmetic progression topology* or *Furstenberg topology* and is denoted by $\mathscr{T}_{ap}$.

8.25. Proposition. *$G \subset \mathbb{Z}$ is $\mathscr{T}_{ap}$-open if and only if $G + a = \{n + a : n \in G\}$ is $\mathscr{T}_{ap}$-open for each $a \in \mathbb{Z}$.*

Proof. We note that the basic open sets in the arithmetic progression topology are nothing but doubly infinite arithmetic sequence. Also, every translate of an arithmetic progression is an arithmetic progression. Thus the proof follows. □

In what follows, $K = \{\frac{1}{n} : n \in \mathbb{N}\}$.

8.26. Example. For $\mathbb{R}$ define

$$\mathscr{B}_K = \{B \subset \mathbb{R} : B = (a, b) \text{ or } B = (a, b) \setminus K, -\infty < a < b < \infty\}.$$

is an open base for a topology on $\mathbb{R}$.

Clearly, $\cup\mathscr{B}_K = \mathbb{R}$. The following table shows the result of intersection of two intersecting elements of $\mathscr{B}_K$.

B_1	B_2	$B_1 \cap B_2$
(a, b)	(c, d)	(e, f)
(a, b)	$(c, d) \setminus K$	$(e, f) \setminus K$
$(a, b) \setminus K$	$(c, d) \setminus K$	$(e, f) \setminus K$

Table 2. $\mathscr{T}_K$-basic open sets intersection

Here $e = \max\{a, c\}$ and $f = \min\{b, d\}$. Thus $\mathscr{B}_K$ is a base for some topology on $\mathbb{R}$.

8.27. Definition. The topology generated by the base $\mathscr{B}_K$ defined in the Example 8.26 is denoted by $\mathscr{T}_K$. $\mathbb{R}$ with $\mathscr{T}_K$ will be denoted by $\mathbb{R}_K$.

8.28. Lemma. *The topologies $\mathscr{T}_\ell$ and $\mathscr{T}_K$ are stronger than the usual topology on $\mathbb{R}$.*

PROOF. Consider the base $\mathscr{B} = \{(a, b) : -\infty < a < b < \infty\}$ for the usual topology $\mathscr{T}$. Since $\mathscr{B} \subset \mathscr{B}_K$, the Corollary 8.22 implies that $\mathscr{T} \subset \mathscr{T}_K$. Now let us take $B = (a, b) \in \mathscr{B}$ and $x \in (a, b)$. Then $B' = [x, b) \subset (a, b)$ and $[x, b) \in \mathscr{B}_\ell$. Thus for each $B \in \mathscr{B}$ and $x \in ellB$ there exists $B' \in \mathscr{B}_\ell$ such that $x \in B' \subset B$. So, by the Lemma 8.21, the topology $\mathscr{T}_\ell$ generated $\mathscr{B}_\ell$ is stronger than the topology $\mathscr{T}$ generated by $\mathscr{B}$. □

Problems

75. Let $\mathscr{B}$ be a base for a topology $\mathscr{T}$ on a set X. Show that

$$\mathscr{T} = \cap\{\mathscr{T}' : \mathscr{T}' \text{ is a topology on } X \text{ and } \mathscr{B} \subset \mathscr{T}'\}.$$

76. Let X be a topological space and $\mathscr{B}_1, \mathscr{B}_2$ be two bases for the topology $\mathscr{T}$ on X. Is $\mathscr{B}_1 \cap \mathscr{B}_2$ a base?

77. Let $\mathscr{B} = \{B \subset \mathbb{R} : B \cap \mathbb{Q} \neq \emptyset\}$. Is $\mathscr{B}$ a base for a topology on $\mathbb{R}$?

78. Let X, Y be two sets and $\mathscr{B}_1$ and $\mathscr{B}_2$ be bases for some topologies on X and Y respectively. Show that

$$\mathscr{B} = \{B_1 \times B_2 : B_1 \in \mathscr{B}, B_2 \in \mathscr{B}_2\}.$$

is a base for some topology on $X \times Y$.

79. Let X be a set and $\mathscr{B}_n = \{G \subset X : |G| \leq n\}$. Show that $\mathscr{B}_n$ is a base for the discrete topology.

80. Which of the following is a base for a topology on $\mathbb{R}$? Compare the so generated topologies.

(1) $\mathscr{B}_1 = \{(a, b) : a, b \in \mathbb{Q}\}$.
(2) $\mathscr{B}_2 = \{(a, b) : a, b \in \mathbb{R} \setminus \mathbb{Q}\}$.
(3) $\mathscr{B}_3 = \{(a, b) : a, b \in \mathbb{Z}\}$.
(4) $\mathscr{B}_4 = \{[a, b) : a, b \in \mathbb{R}\}$.
(5) $\mathscr{B}_5 = \{[a, b) : a, b \in \mathbb{Q}\}$.
(6) $\mathscr{B}_6 = \{[a, b) : a, b \in \mathbb{R} \setminus \mathbb{Q}\}$.
(7) $\mathscr{B}_7 = \{[a, b) : a, b \in \mathbb{Z}\}$.
(8) $\mathscr{B}_8 = \{(a, \infty) : a \in \mathbb{R}\}$.
(9) $\mathscr{B}_9 = \{(a, \infty) : a \in \mathbb{Q}\}$.
(10) $\mathscr{B}_{10} = \{(a, \infty) : a \in \mathbb{R} \setminus \mathbb{Q}\}$.
(11) $\mathscr{B}_{11} = \{(a, \infty) : a \in \mathbb{Z}\}$.
(12) $\mathscr{B}_{12} = \{[a, \infty) : a \in \mathbb{R}\}$.
(13) $\mathscr{B}_{13} = \{[a, \infty) : a \in \mathbb{Q}\}$.
(14) $\mathscr{B}_{14} = \{[a, \infty) : a \in \mathbb{R} \setminus \mathbb{Q}\}$.
(15) $\mathscr{B}_{15} = \{[a, \infty) : a \in \mathbb{Z}\}$.
(16) $\mathscr{B}_{16} = \{[a, b] : a \in \mathbb{Q}, b \in \mathbb{R} \setminus \mathbb{Q}\}$.
(17) $\mathscr{B}_{17} = \{[a, b] : a \in \mathbb{Z}, b \in \mathbb{Q} \setminus \mathbb{Z}\}$.

Frame the analogous question by replacing $[a, b)$, (a, ∞) and $[a, \infty)$, with the intervals $(a, b]$, $(-\infty, a)$ and $(-\infty, a]$. Also, frame other set of questions by taking both the endpoints from the different sets rather than taking both endpoints from the same sets $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{R} \setminus \mathbb{Q}$.

81. Let X be a non-empty set. Check if any of the following collections of subsets of X forms a topology on X.

(1) $\mathscr{T}_c = \{U \subset X \mid U = \emptyset \text{ or } X \setminus U \text{ is countable}\}$.
(2) $\mathscr{T}_\infty = \{U \subset X \mid X \setminus U \text{ is infinite}\} \cup \{\emptyset, X\}$.

(3) $\mathscr{T} = \{U \subset X \mid U \text{ is finite}\}$.
(4) $\mathscr{T} = \{U \subset X \mid U \text{ is infinite}\}$.

82. Show that the usual topology is weaker than the lower limit topology as well as the upper limit topology on $\mathbb{R}$.

83. Show that $\mathscr{T}_\ell \cap \mathscr{T}_u$ is the usual topology on $\mathbb{R}$.

84. Let X_1, X_2 be two topological spaces and X be a set. Let $f\colon X \to X_1$ and $g\colon X \to X_2$ be two functions. Define

$$\mathscr{B} = \{f^{-1}(U) \cap g^{-1}(V) : U \text{ is open in } X_1 \text{ and } V \text{ is open in } X_2\}.$$

Show that $\mathscr{B}$ is a base for some topology on X.

85. Let X be a set, $(Y, \mathscr{T})$ be a topological space, $f\colon X \to Y$ be a function and $\mathscr{B}$ be a base for $\mathscr{T}$. Show that $\mathscr{B}_w(f) = \{f^{-1}(B) : B \in \mathscr{B}\}$ is a base for a topology on X.

9. Subbase

9.1. Definition. Let $(X, \mathscr{T})$ be a topological space. A collection $\mathscr{S} \subset \mathscr{T}$ is said to be a *subbase* or a *subbasis* for $\mathscr{T}$ if

$$\{\cap\mathscr{C} : \mathscr{C} \text{ is a finite subset of } \mathscr{S}\}$$

is a base for $\mathscr{T}$. In context of $\mathscr{S}$, an element of $\mathscr{S}$ is called a *subbasic open set* and $\mathscr{T}$ is called the *topology generated by* $\mathscr{S}$.

9.2. Lemma. *Let X be a nonempty set. Then every collection of subsets of X is a subbase for some topology on X.*

Proof. Let $\mathscr{S} \subset \mathscr{P}(X)$. Define

$$\mathscr{B} = \{B \subset X : B \text{ is a finite intersection of members of } \mathscr{S}\}.$$

Let $\mathscr{C} = \emptyset$, the empty family of members of $\mathscr{S}$. Then $\cap\mathscr{C} = X$. So, $X \in \mathscr{B}$.

Now let $B_1, B_2 \in \mathscr{B}$ and $x \in B_1 \cap B_2$. Then $B_1 = S_{11} \cap S_{12} \cap S_{13} \cap \cdots \cap S_{1k}$ and $B_2 = S_{21} \cap S_{22} \cap S_{23} \cap \cdots \cap S_{2m}$. With $S_{ij} \in \mathscr{S}$ for all i, j. But then

$$B_1 \cap B_2 = S_{11} \cap S_{12} \cap S_{13} \cap \cdots \cap S_{1k} \cap S_{21} \cap S_{22} \cap S_{23} \cap \cdots \cap S_{2m}$$

is also a finite intersection of members of $\mathscr{S}$.[3] So, $B_1 \cap B_2 \in \mathscr{B}$. So, by the Corollary 8.6, $\mathscr{B}$ is a base for some topology on X. □

9.3. Proposition. *Let X be a nonempty set and $\mathscr{S}$ be a subbase for the topology $\mathscr{T}$. Then $\mathscr{T}$ is the smallest topology on X such that $\mathscr{S} \subset \mathscr{T}$.*

Proof. Let $\mathscr{T}'$ be another topology on X such that $\mathscr{S} \subset \mathscr{T}'$. Let $\mathscr{B}$ be the base generated by $\mathscr{S}$ for $\mathscr{T}$. Let $B \in \mathscr{B}$. Then $B = \cap\mathscr{C}$ for some finite subfamily of $\mathscr{C} \subset \mathscr{S} \subset \mathscr{T}'$. Thus B is a finite intersection of $\mathscr{T}'$-open sets. Thus B is $\mathscr{T}'$-open

[3] This phenomenon can be described as *a finite intersection of finite intersections is a finite intersection.*

set. That is, $B \in \mathscr{T}'$. Hence $\mathscr{B} \subset \mathscr{T}'$. But then by the Proposition 8.2, $\mathscr{T}$ is the smallest topology containing $\mathscr{B}$. Hence $\mathscr{T} \subset \mathscr{T}'$. This completes the proof. □

Problems

86. Let X be a set and $\mathscr{S}$ be a subbase for a topology $\mathscr{T}$. Show that $\mathscr{T}$ is the weakest topology such that $\mathscr{S} \subset \mathscr{T}$.

87. Show that $\mathscr{S} = \{(a, b) \subset \mathbb{R} : \text{either } a = -\infty \text{ or } b = \infty\}$ is a subbase for the usual topology on $\mathbb{R}$. Also, show that it is not a base.

88. Show that $\mathscr{S} = \{[a, \infty) \subset \mathbb{R} : a \in \mathbb{R}\} \cup \{(-\infty, b) \subset \mathbb{R} : b \in \mathbb{R}\}$ is a subbase for the lower limit topology on $\mathbb{R}$.

89. Let X be a set. Show that the following are subbases for the discrete topology on X.

(1) If X has at least four elements, then the collection of all subsets of X having exactly three elements.
(2) The collection of all countable subsets of X.
(3) The collection of all finite subsets of X.
(4) If X is infinite, then the collection of all countably infinite subsets of X.

90. Let X be a set and

$$\mathscr{S} = \{V \subset X : X \setminus V \text{ is singleton}\}.$$

Show that $\mathscr{S}$ is a subbase for the cofinite topology on X.

10. Neighbourhoods and Local Base

A local base is associated to a point rather than the whole of the topology. We begin with the following definition of a neighbourhood of a point in a topological space.

10.1. Definition. Let $(X, \mathscr{T})$ be a topological space and $x \in X$. $U \subset X$ is called a *neighbourhood of x* if there is an open set G such that $x \in G \subset U$. We set the notation

$$\mathscr{U}_x = \{U \subset X : U \text{ is a neighbourhood of } x\}.$$

The family $\mathscr{U}_x$ is called *the neighbourhood system* at x.

10.2. Definition. Let $(X, \mathscr{T})$ be a topological space and $x \in X$. A family $\mathscr{B}_x$ of neighbourhoods of x is said to be a *local base at x* or a *neighbourhood base at x* if for every neighbourhood U of x, there is $B \in \mathscr{B}_x$ such that $B \subset U$.

Unlike members of a base, a member of a local base need not be open.

10.3. Example. Let X be a topological space. Then for every $x \in X$, $\mathscr{U}_x$ is a local base at x.

10.4. Example. Let X be a discrete topological space. Then a collection $\mathscr{B}_x$ of neighbourhoods of x is a local base at x if and only if $\{x\} \in \mathscr{B}_x$.

Indeed, suppose $\{x\} \in \mathscr{B}_x$. Then for every subset $A \subset X$ containing x, we have $\{x\} \in \mathscr{B}_x$ such that $\{x\} \subset A$.

On the other hand, since $\{x\}$ is a neighbourhood of x, there must be $B \in \mathscr{B}_x$ such that $x \in B \subset \{x\}$. Thus $B = \{x\}$. Thus $\{x\} \in \mathscr{B}_x$.

10.5. Proposition. *Let X be a topological space and for every $x \in X$, let $\mathscr{B}_x$ be a local base at x. Then $G \subset X$ is open if and only if for every $x \in G$, there is $B_x \in \mathscr{B}_x$ such that $B_x \subset G$.*

Proof. Suppose G is open. Let $x \in G$. Then G is a neighbourhood of x. Hence there is $B_x \in \mathscr{B}_x$ such that $B_x \subset B$.

Conversely, suppose for every $x \in G$, there is $B_x \in \mathscr{B}_x$ such that $B_x \subset G$. Let $x \in G$. Choose some $B_x \in \mathscr{B}_x$ such that $B_x \subset G$. Since $B_x \in \mathscr{B}_x$, B_x is a neighbourhood of x. Hence there is some open set U_x such that $x \in U_x \subset B_x$. Thus for every $x \in G$, we have an open set U_x such that $x \in U_x \subset G$. Taking union of all such U_x, we get $\cup_{x\in G} U_x = G$. Thus G is open. This completes the proof. □

Our first goal is to characterize topology in terms of local bases.

10.6. Theorem. *Let X be a topological space. Then the neighbourhood system is the system of families $\mathscr{U}_x$, assigned to $x \in X$, satisfies the following.*

(1) $U \in \mathscr{U}_x \Rightarrow x \in U$.
(2) $U, V \in \mathscr{U}_x \Rightarrow U \cap V \in \mathscr{U}_x$.
(3) For each $x \in X$ and for every $U \in \mathscr{U}_x$, there is $V \in \mathscr{U}_x$ such that $U \in \mathscr{U}_y$ for every $y \in V$.
(4) $U \in \mathscr{U}_x, U \subset V \Rightarrow V \in \mathscr{U}_x$.
(5) $G \subset X$ is open if and only if $G \in \mathscr{U}_x$ for every $x \in G$.

Proof. ☞(1) follows from the definition of a neighbourhood.

☞(2) Let $U, V \in \mathscr{U}_x$. So, there are open sets G, H such that $x \in G \subset U$ and $x \in H \subset V$. Thus $G \cap H$ is an open set such that $x \in G \cap H \subset U \cap V$. Thus $U \cap V \in \mathscr{U}_x$.

☞(3) Let $U \in \mathscr{U}_x$. So, there is an open set V such that $x \in V \subset U$. Then for every $y \in V$, V is an open set such that $y \in V \subset U$. Thus $U \in \mathscr{U}_y$ for every $y \in V$.

☞(4) Let $U \in \mathscr{U}_x$ and $U \subset V$. Then there is an open set G such that $x \in G \subset U \subset V$. Thus $V \in \mathscr{U}_x$.

☞(5) Let G be open. Then for every $x \in G$, we have $x \in G \subset G$. Thus $G \in \mathscr{U}_x$.

Conversely, suppose $G \in \mathscr{U}_x$ for every $x \in G$. So, for every $x \in G$, there is an open set U_x such that $x \in U_x \subset G$. Thus $G = \cup_{x\in G} U_x$, a union of open sets, and hence, G is an open set. □

The following constructs a topology from a neighbourhood system and also characterizes families which are neighbourhood systems for some topologies.

10.7. Theorem. *Let X be a set. Suppose there is a system of nonempty families $\mathscr{U}^x$ assigned to each $x \in X$ satisfying the following.*

(1) $U \in \mathscr{U}^x \Rightarrow x \in U$.
(2) $U, V \in \mathscr{U}^x \Rightarrow U \cap V \in \mathscr{U}^x$.
(3) For each $x \in X$ and for every $U \in \mathscr{U}^x$, there is $V \in \mathscr{U}^x$ such that $U \in \mathscr{U}^y$ for every $y \in V$.
(4) $U \in \mathscr{U}^x, U \subset V \Rightarrow V \in \mathscr{U}^x$.

Define

$$\mathscr{T} = \{G \subset X : G \in \mathscr{U}^x \text{ for every } x \in G\}.$$

Then $\mathscr{T}$ is a topology on X and $\mathscr{U}^x = \mathscr{U}_x$ for all $x \in X$.

Proof. Vacuously, $\emptyset \in \mathscr{U}_x$ for every $x \in \emptyset$. Thus $\emptyset \in \mathscr{T}$. Let $x \in X$. Since $\mathscr{U}^x \neq \emptyset$, there is $U \in \mathscr{U}^x$. But then by (4), $X \in \mathscr{U}^x$. Hence $X \in \mathscr{T}$.

Let $\{G_i : i \in I\} \subset \mathscr{T}$. Let $x \in \cup_{i\in I} G_i$. So there is $j \in I$ such that $x \in G_j$. But then $G_j \in \mathscr{U}^x$, and hence, by (4), $\cup_{i\in I} G_i \in \mathscr{U}^x$. So, $\cup_{i\in I} G_i \in \mathscr{T}$.

Let $G, H \in \mathscr{T}$. Let $x \in G \cap H$. So, $x \in G$ and $x \in H$. But then $G, H \in \mathscr{U}^x$. So by (2), $G \cap H \in \mathscr{U}^x$ for every $x \in G \cap H$. So, $G \cap H \in \mathscr{T}$. So, the proof till this point establishes the fact that $\mathscr{T}$ is a topology on X.

Let $x \in X$ and $U \in \mathscr{U}^x$. Define $V = \{y \in U : U \in \mathscr{U}^y\}$. We show that V is open. Let $y \in V$. So, $U \in \mathscr{U}^y$. Hence there is $W \in U^y$ such that $U \in U^z$ for every $z \in W$. Thus $W \subset V$. Thus $y \in W \subset V$ and $W \in U^y$. Thus $V \in \mathscr{U}^y$. Hence $V \in \mathscr{U}^y$ for every $y \in V$. Thus $V \in \mathscr{T}$. But $V \subset U$. Hence $U \in \mathscr{U}_x$. Thus $\mathscr{U}^x \subset \mathscr{U}_x$. Conversely, suppose $U \in \mathscr{U}_x$. Let $G = U^\circ$. Then $x \in G \subset U$. Since $G \in \mathscr{T}$, $G \in \mathscr{U}^x$. But $G \subset U$. Hence $U \in \mathscr{U}^x$. This shows that $\mathscr{U}_x \subset \mathscr{U}^x$, which completes the proof. □

10.8. Theorem. *Let X be a topological space and for each $x \in X$, let $\mathscr{B}_x$ be a neighbourhood base at x. Then the following hold.*

(1) If $U \in \mathscr{B}_x$, then $x \in U$.
(2) If $U_1, U_2 \in \mathscr{B}_x$, then there is $U_3 \in \mathscr{B}_x$ such that $U_3 \subset U_1 \cap U_2$.
(3) For every $U \in \mathscr{B}_x$, there is $V \in \mathscr{B}_x$ such that for every $y \in V$, there is $W \in \mathscr{B}_y$ such that $W \subset U$.

Conversely, suppose X is a set and assigned to each $x \in X$, there is a family $\mathscr{B}_x$ of subsets of X satisfying (1)–(3). Define

(4) $\mathscr{T} = \{G \subset X :$ for every $x \in G$, there is $U \in \mathscr{B}_x$ such that $U \subset G\}$.

Then $\mathscr{T}$ is a topology on X and $\mathscr{B}_x$ is a neighbourhood base at x.

Proof. If $\mathscr{B}_x$ is a neighbourhood base at each $x \in X$, then (1), (2) follow from the definition and the fact that intersection of two neighbourhoods of x is again a neighbourhood of x. Now let $U \in \mathscr{B}_x$. So, $U \in \mathscr{U}_x$. Let G be an open set such that $x \in G \subset U$. But G is a neighbourhood of x. Hence there is $V \in \mathscr{B}_x$ such that $x \in V \subset G$. Now let $y \in V$. Then $y \in V \subset G$. Thus $G \in \mathscr{U}_y$. Hence there is

$W \in \mathscr{B}_y$ such that $y \in W \subset G \subset U$. Thus for every $y \in V$, we get $W \in \mathscr{B}_y$ such that $W \subset U$. This proves (3).

Conversely, suppose, X is a set and assigned to each $x \in X$, there is a family $\mathscr{B}_x$ of subsets of X satisfying (1)–(3). Define $\mathscr{T}$ by (4). Clearly, $\emptyset, X \in \mathscr{T}$. Let $\{G_i : i \in I\} \subset \mathscr{T}$ and $x \in G = \cup_{i\in I} G_i$. Then $x \in G_j$ for some $j \in I$. But then there is $U \in \mathscr{B}_x$ such that $U \subset G_j \subset G$. Thus $\cup_{i\in I} G_i \in \mathscr{T}$. Now let $G_1, G_2 \in \mathscr{T}$ and $x \in G_1 \cap G_2$. Then there are $U_1, U_2 \in \mathscr{B}_x$ such that $U_1 \subset G_1, U_2 \subset G_2$. Using (2), let $U_3 \in \mathscr{B}_x$ be such that $U_3 \subset U_1 \cap U_2$. But then $x \in U_3 \subset U_1 \cap U_2 \subset G_1 \cap G_2$. This shows that $G_1 \cap G_2 \in \mathscr{T}$ and also completes the proof. □

10.9. Proposition. *Let X be a topological space and $\mathscr{B} \subset \mathscr{T}$. For every $x \in X$ define $\mathscr{B}_x = \{B \in \mathscr{B} : x \in B\}$. Then $\mathscr{B}$ is a base for $\mathscr{T}$ if and only if each $\mathscr{B}_x$ is a local base at x.*

Proof. Let $\mathscr{B}$ be a base for $\mathscr{T}$ and $x \in X$. For $U \in \mathscr{U}_x$, there is $B \in \mathscr{B}$ such that $x \in B \subset U$. But then we get $B \in \mathscr{B}_x$ and $B \subset U$. Thus $\mathscr{B}_x$ is a base at x.

Conversely, suppose each $\mathscr{B}_x$ is a local base at x. Let $U \in \mathscr{T}$. Then for each $x \in U$, there is $B_x \in \mathscr{B}_x \subset \mathscr{B}$ such that $B_x \subset U$. That is, $U = \cup_{x\in U} B_x$. Thus U is a union of members of $\mathscr{B}$. Thus $\mathscr{B}$ is a base for $\mathscr{T}$. □

11. Countability Axioms

11.1. Definition. A topological space is said to be *first countable* if there is a countable local base B_x at every $x \in X$. A topological space is said to be *second countable* if it has a countable base.

11.2. Example. $\mathbb{R}$ is first countable.

Indeed for every $x \in X$, $\mathscr{B}_x = \{(x - \frac{1}{n}, x + \frac{1}{n}) : n \in \mathbb{N}\}$ is a local base at x.

11.3. Theorem. *Every second countable space is first countable.*

Proof. Let X be a second countable space. So, it has a countable base $\mathscr{B}$. Then by the Proposition 10.9, $\mathscr{B}_x = \{B \in \mathscr{B} : x \in B\}$ is a local base at x for every $x \in X$. Since $\mathscr{B}_x \subset \mathscr{B}$, $\mathscr{B}_x$ is countable. This completes the proof. □

11.4. Example. A first countable topological space need not be second countable.

Indeed, for the discrete space $\mathbb{R}$, $\{\{x\}\}$ is a local base at x for every $x \in \mathbb{R}$. However, any base for a discrete space must contain all the singletons. Thus a base for the discrete topology on $\mathbb{R}$ cannot be countable.

11.5. Example.

(1) $\{(a, b) : a, b \in \mathbb{Q}\}$ is a countable base for the usual topology on $\mathbb{R}$. Thus $\mathbb{R}$ is second countable.

(2) $\{[x, x+\frac{1}{n}) : n \in \mathbb{N}\}$ is a local base at x for the lower limit topology. Thus $\mathbb{R}_\ell$ is first countable. But $\mathbb{R}_\ell$ is not second countable. Suppose $\mathscr{B} = \{B_n : n \in \mathbb{N}\}$

is a base for $\mathscr{T}_\ell$. Without loss of generality, we assume that $B_n \neq \emptyset$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, define $a_n = \inf B_n$ and $A = \{a_n : n \in \mathbb{N}\}$. Clearly, A is countable. It may contain $-\infty$ is some B_n is not bounded below. Let $a \in \mathbb{R}$. Consider $[a, b) \in \mathscr{T}_\ell$. Since $a \in [a, b)$, there is $B_n \in \mathscr{B}$ such that $a \in B_n \subset [a, b)$. Since $B_n \subset [a, b)$, $a = \inf[a, b) \leq \inf B_n$. Also, since $a \in B_n$, $\inf B_n \leq a$. Thus $\inf B_n = a$. That is $a \in A$, and hence, $\mathbb{R} \subset A$. This is a contradiction to the fact that $\mathbb{R}$ is uncountable. Thus $\mathbb{R}_\ell$ is not second countable.

(3) $\{(x - \frac{1}{n}, x] : n \in \mathbb{N}\}$ is a local base at x for the upper limit topology. Thus $\mathbb{R}_u$ is first countable. Replacing inf by sup in the previous proof, we conclude that $\mathbb{R}_u$ is not second countable.

(4) $\{X\}$ is a local base at each $x \in X$ for the indiscrete topological space. It is a base too. Thus the indiscrete spaces are second countable.

(5) A finite space has only finitely many open sets and since the topology itself is a base, the space is second countable. So, it is also a first countable.

11.6. Proposition. *A cofinite space $(X, \mathscr{T}_f)$ is first countable if and only if X is countable.*

Proof. Suppose $X = \{x_1, x_2, x_3, \dots\}$, a countable set. Note that the open subsets of X containing a fixed point $x_i \in X$ are in one to one correspondence with all finite subsets of $X \setminus \{x_i\}$. Since by Proposition 3.18 (4), the number of finite subsets of a countable sets is countable, we conclude that $\mathscr{U}_{x_i}$ is countable. Thus X is first countable.

Now suppose that X is uncountable. Let $x \in X$. Suppose, if possible, there is a countable local base $\{B_n : n \in \mathbb{N}\}$ at x. Indeed, $X \setminus B_n$ is a finite set for every n, and hence, $X \setminus \cap B_n = \cup(X \setminus B_n)$ is countable. Hence $\cap B_n$ must be uncountable. Let $y \in \cap B_n \setminus \{x\}$. But then every open set containing x also contains y. On the other hand, $X \setminus \{y\}$ is an open set contains x but not y, a contradiction. Thus X is not first countable. □

11.7. Proposition. *A cofinite space $(X, \mathscr{T}_f)$ is second countable if and only if X is countable.*

Proof. Suppose $X = \{x_1, x_2, x_3, \dots\}$, a countable set. Note that the nonempty open subsets of X are in one to one correspondence with all finite subsets of X. Since by Proposition 3.18 (4), the number of finite subsets of a countable sets is countable, we conclude that $\mathscr{T}_f$ is countable. Thus X is second countable.

Now suppose that X is uncountable. By Proposition 11.6, $(X, \mathscr{T}_f)$ is not first countable. Hence by Theorem 11.3, X is not second countable. □

Problems

91. Show that the cofinite topology on $\mathbb{R}$ is not first countable.

92. Show that the cocountable topology on $\mathbb{R}$ is not first countable.

93. Show that for a topological space $(X, \mathscr{T})$, if either X or $\mathscr{T}$ is finite, then show that $(X, \mathscr{T})$ is second countable.

12. Subspaces

Let $(X, \mathscr{T})$ be a topological space and $Y \subset X$. We define the subspace topology on Y as follows.

12.1. Proposition. *Let $(X, \mathscr{T})$ be a topological space and $Y \subset X$. Define*

$$\mathscr{T}_Y = \{G \cap Y : G \in \mathscr{T}\}. \tag{12.1.1}$$

Then $\mathscr{T}_Y$ is a topology on Y.

Proof. Clearly, $\emptyset = \emptyset \cap Y$ and $Y = X \cap Y$ belong to $\mathscr{T}_Y$.

For a family $\{G_i \cap Y : i \in I\} \subset \mathscr{T}_Y$ with $\{G_i : i \in I\} \subset \mathscr{T}$, since $\cup(G_i \cap Y) = (\cup G_i) \cap Y$ with $\cup G_i \in \mathscr{T}$, we see that $\cup(G_i \cap Y) \in \mathscr{T}_Y$.

For $G \cap Y, H \cap Y \in \mathscr{T}_Y$ with $G, H \in \mathscr{T}$, $G \cap Y \cap H \cap Y = G \cap H \cap Y \in \mathscr{T}_Y$ because $G \cap H \in \mathscr{T}$. Thus $\mathscr{T}_Y$ is a topology on Y. □

12.2. Definition. Let $(X, \mathscr{T})$ be a topological space and $Y \subset X$. The topology $\mathscr{T}_Y$ defined in (12.1.1) is called the subspace topology on Y. Also, $(Y, \mathscr{T}_Y)$ is called a *subspace* of $(X, \mathscr{T})$.

12.3. Proposition. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{B}$ be a base for $\mathscr{T}$. Let $Y \subset X$. Then $\mathscr{B}_Y = \{B \cap Y : B \in \mathscr{B}\}$ is base for $\mathscr{T}_Y$.*

Proof. Clearly, for $B \in \mathscr{B}$, $B \in \mathscr{T} \Rightarrow B \cap Y \subset \mathscr{T}_Y$. Thus $\mathscr{B}_Y \subset \mathscr{T}_Y$. Now let $G \cap Y \in \mathscr{T}_Y$ with $G \in \mathscr{T}$. Then $G = \cup i \in I B_i$ with $B_i \in \mathscr{B}$ for every $i \in I$. But then $G \cap Y = \cup_{i \in I}(B_i \cap Y)$, with $B_i \cap Y \in \mathscr{B}_Y$ for every $i \in I$. Thus $\mathscr{B}_Y$ is a base for $\mathscr{T}_Y$. □

An argument similar to the above proposition can prove the following.

12.4. Proposition. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{S}$ be a base for $\mathscr{T}$. Let $Y \subset X$. Then $\mathscr{S}_Y = \{S \cap Y : S \in \mathscr{S}\}$ is subbase for $\mathscr{T}_Y$.*

12.5. Proposition. *Let X be a topological space, $x \in Y \subset X$ and $\mathscr{B}_x$ be a local base at x in X. Then $\mathscr{B}_{xY} = \{B \cap Y : B \in \mathscr{B}_x\}$ is a local base at x in Y.*

In general, an open subset of a subspace need not be open in the main topological space. Indeed, let $X = \mathbb{R}$ and $Y = [0, 1]$. Then Y is open in Y but Y is not open in X. This may be true even for a proper subset of a subspace too. Observe that $(0.1, 1]$ is open in Y but not in X. The following asserts that *open in open is open*; that is, open subset of an open subspace is an open subset.

12.6. Proposition. *Let $(X, \mathscr{T})$ be a topological space and Y be an open subset of X. Then $A \subset Y$ is open in X if and only if it is open in Y.*

Proof. Let A be open in X. Then $A = A \cap Y$ is open in Y. Conversely, suppose A is open in Y. So, by the definition of $\mathscr{T}_Y$, there is a $\mathscr{T}$-open subset G of X such that $A = G \cap Y$. Since G and Y are $\mathscr{T}$-open, so is A. □

In passing, we define the following, that will be useful in the discussion of limit of a function.

12.7. Definition. Let $(X, \mathscr{T})$ be a topological space and $Y \subset X$. A point $y \in Y$ is said to be an isolated point of Y if $\{y\}$ is open in Y.

For a subset Y of a topological space X, some authors prefer an equivalent definition of an isolated point of Y as a point $y \in Y$ for which there is a neighbourhood U in X such that $U \cap Y = \emptyset$.

12.8. Example.

(1) Subspace of a discrete space is discrete.
(2) Subspace of an indiscrete space is indiscrete.
(3) Subspace of a cofinite space is cofinite.
(4) Subspace of a cocountable space is cocountable.
(5) $\mathbb{Z}$ carries the discrete topology as a subspace of $\mathbb{R}$ with the usual topology as well as the discrete topology.

Let $Z \subset Y \subset X$. It seems that Z carries two topologies, one as a subspace of Y and the other as a subspace of X. We show in the following proposition that they are equal.

12.9. Proposition. *For a topological space X and $Z \subset Y \subset X$, $\mathscr{T}_Z = (\mathscr{T}_Y)_Z$ on Z.*

Proof. Let $G \in \mathscr{T}_Z$. So, there is $H \in \mathscr{T}$ such that $G = H \cap Z$. Consequently, $G = H \cap Z = H \cap Y \cap Z = (H \cap Y) \cap Z \in (\mathscr{T}_Y)_Z$ because $H \cap Y \in \mathscr{T}_Y$.

Conversely, suppose $G \in (\mathscr{T}_Y)_Z$. So, there is $K \in \mathscr{T}_Y$ such that $G = K \cap Z$. But $K \in \mathscr{T}_Y$ implies the existence of $H \in \mathscr{T}$ such that $K = H \cap Y$. Thus $G = K \cap Z = (H \cap Y) \cap Z = H \cap Y \cap Z = H \cap Z$. Thus $G \in \mathscr{T}_Z$. □

We turn now to extend a topology from a subset to a set. Before we embark a general case, we discuss particular cases.

12.10. Proposition. *Let X be a set, $Y \subset X$ and $\mathscr{T}$ be a topology on Y. Define*

$$\mathscr{T}^{Y\mathscr{D}} = \{G \subset X : G \cap Y \text{ is open in } Y\}.$$

Then $\mathscr{T}^{Y\mathscr{D}}$ is a topology on X, $\mathscr{T} = {\mathscr{T}^{Y\mathscr{D}}}_Y$ and ${\mathscr{T}^{Y\mathscr{D}}}_{X\setminus Y}$ is discrete.

PROOF. Clearly, $\emptyset \cap Y$ and $X \cap Y$ are $\mathscr{T}$-open. Hence $\emptyset$ and X are in $\mathscr{T}^{Y\mathscr{D}}$. For $\{G_i : i \in I\} \subset \mathscr{T}^{Y\mathscr{D}}$, we have $G_i \cap Y \in \mathscr{T}$ for every $i \in I$, and hence, $(\cup G_i) \cap Y = \cup_i (G_i \cap Y) \in \mathscr{T}$ because $\mathscr{T}$ is a topology. Thus $\cup_i G_i \in \mathscr{T}^{Y\mathscr{D}}$. Also, for $G, H \in \mathscr{T}^{Y\mathscr{D}}$, $G \cap Y, H \cap Y \in \mathscr{T}$. Hence $(G \cap H) \cap Y = (G \cap Y) \cap (H \cap Y) \in \mathscr{T}$. Thus $G \cap H \in \mathscr{T}^{Y\mathscr{D}}$. It follows from the definition of $\mathscr{T}^{Y\mathscr{D}}$ that $\mathscr{T} = \mathscr{T}^{Y\mathscr{D}}{}_Y$. Now let $G \subset X \setminus Y$. Then $G \cap Y = \emptyset \in \mathscr{T}$. Consequently, $G \in \mathscr{T}^{Y\mathscr{D}}$, and hence, $G = G \cap (X \setminus Y) \in \mathscr{T}^{Y\mathscr{D}}{}_{X\setminus Y}$. So, $X \setminus Y$ is discrete. □

12.11. Proposition. *Let X be a set, $Y \subset X$ and $\mathscr{T}$ be a topology on Y. Define*

$$\mathscr{T}^{Y\mathscr{I}} = \{G \subset X : G \in \mathscr{T}_1 \text{ or } G = X\}.$$

Then $\mathscr{T}^{Y\mathscr{I}}$ is a topology on X, $\mathscr{T} = \mathscr{T}^{Y\mathscr{I}}$ and $\mathscr{T}^{Y\mathscr{I}}{}_{X\setminus Y}$ is indiscrete.

PROOF. By the definition of $\mathscr{T}^{Y\mathscr{I}}$, $\emptyset, X \in \mathscr{T}^{Y\mathscr{I}}$. Let $\{G_i : i \in I\} \subset \mathscr{T}^{Y\mathscr{I}}$. If $G_i = X$ for some $i \in I$, then $\cup_{i\in I} G_i = X \in \mathscr{T}^{Y\mathscr{I}}$. Otherwise, $G_i \in \mathscr{T}$ for every $i \in I$. Consequently, $\cup_{i\in I} G_i \in \mathscr{T}$, and hence, $\cup_{i\in I} G_i \in \mathscr{T}^{Y\mathscr{I}}$. Now let $G, H \in \mathscr{T}^{Y\mathscr{I}}$. If $G = H = X$, we are done. If $G = X \neq H$, then $H \in \mathscr{T}$ and so, $G \cap H = X \cap H = H \in \mathscr{T}$. If $G \neq X \neq H$, then $G, H \in \mathscr{T}$, giving $G \cap H \in \mathscr{T}$. Thus in each case, $G \cap H \in \mathscr{T}^{Y\mathscr{I}}$. This shows that $\mathscr{T}$ is a topology on X. Also for $G \in \mathscr{T}^{Y\mathscr{I}}$, if $G = X$, then $G \cap Y = Y$, otherwise $G \cap Y = G \in \mathscr{T}$. Thus $\mathscr{T} = \mathscr{T}^{Y\mathscr{I}}{}_Y$. On the other hand, for $G \in \mathscr{T}$, $G \cap (X \setminus Y) = \emptyset$ and $X \cap (X \setminus Y) = X \setminus Y$. Thus $\mathscr{T}^{Y\mathscr{I}}{}_{X\setminus Y}$ is indiscrete. □

Both of the above propositions will follow as the particular cases of the following theorem.

12.12. Theorem. *Let X be a set. Let $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \emptyset$. For $k = 1, 2$, let $\mathscr{T}_k$ be a topology on X_k. Define*

$$\mathscr{T} = \{G \subset X : G \cap X_k \in \mathscr{T}_k \text{ for } k = 1, 2\}.$$

Then $\mathscr{T}$ is a topology on X and $\mathscr{T}_{X_k} = \mathscr{T}_k$, $(k = 1, 2)$.

PROOF. Given $G \subset X$, we fix the notation $G_k = G \cap X_k$, $k = 1, 2$. In this case, $G = G_1 \cup G_2$. Also, $G \in \mathscr{T}$ if and only if $G_k \in \mathscr{T}_k$ for $k = 1, 2$. Clearly, for $k = 1, 2$, $\emptyset = \emptyset \cap X_k \in \mathscr{T}_k$ and $X_k = X \cap X_k \in \mathscr{T}_k$. Thus $\emptyset, X \in \mathscr{T}$. Let $\{G_i : i \in I\} \subset \mathscr{T}$. Then $(\cup_{i\in I} G_i)_k = (\cup_{i\in I} G_i) \cap X_k = \cup_{i\in I}(G_i \cap X_k) = \cup_{i\in I} G_{ik} \in \mathscr{T}_k$ for every $k = 1, 2$.

Now let $G, H \in \mathscr{T}$. Then $(G \cap H)_k = (G \cap H) \cap X_k = (G \cap X_k) \cap (H \cap X_k) = (G_k \cap H_k) \in \mathscr{T}_k$, $(k = 1, 2)$. Thus $\mathscr{T}$ is a topology on X.

Now let $G \in \mathscr{T}_1$. Then $G \cap X_1 = G$ and $G \cap X_2 = \emptyset$. Thus $G \in \mathscr{T}$, and so, $G = G \cap X_1 \in \mathscr{T}_{X_1}$. Conversely, suppose $G \in \mathscr{T}_{X_1}$. So, there is $G' \in \mathscr{T}$ such that $G = G' \cap X_1$. But then $G \in \mathscr{T}_1$. Thus $\mathscr{T}_1 = \mathscr{T}_{X_1}$. In the same way, $\mathscr{T}_2 = \mathscr{T}_{X_2}$. This completes the proof. □

It is at once that Proposition 12.10 follows by endowing the discrete topology to $X \setminus Y$ and Proposition 12.11 follows by endowing the indiscrete topology to $X \setminus Y$. Consequently, we have the following definition.

12.13. Definition. Let X be a set, $Y \subset X$ and $\mathscr{T}$ be a topology on Y. The topology $\mathscr{T}^{Y\mathscr{D}}$ defined in Proposition 12.10 is called the *discrete extension of $\mathscr{T}$ to X* and the topology $\mathscr{T}^{Y\mathscr{I}}$ defined in Proposition 12.11 is called the *indiscrete extension of $\mathscr{T}$ to X*.

Problems

94. Prove the Proposition 12.4.

95. Prove the Proposition 12.5.

96. Let Y be a subspace of a topological space X and $y \in Y$. Show that y is an isolated point of Y if and only if y is not a limit point of Y.

97. Consider the set $Y = [-1, 1]$ as a subspace of $\mathbb{R}$. Which of the following sets are open in Y and which of them are open in $\mathbb{R}$?

(1) $A = \{x : \frac{1}{2} < |x| < 1\}$.
(2) $A = \{x : \frac{1}{2} < |x| \leq 1\}$.
(3) $A = \{x : \frac{1}{2} \leq |x| < 1\}$.
(4) $A = \{x : \frac{1}{2} \leq |x| \leq 1\}$.
(5) $A = \{x : 0 < |x| < 1 \text{ and } \frac{1}{x} \notin \mathbb{N}\}$.

13. Closed Sets

13.1. Definition. Let $(X, \mathscr{T})$ be a topological space. A subset F of X is said to be *closed in X* (or $\mathscr{T}$-closed) if $X \setminus F$ is open in X.

13.2. Examples.

(1) $[a, b]$ is closed in the usual topology on $\mathbb{R}$. Indeed, $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$, being a union of two open intervals, is open in $\mathbb{R}$.
(2) A singleton is closed in the usual topology on $\mathbb{R}$. Indeed for $x \in \mathbb{R}$, $(-\infty, x) \cup (x, \infty)$, being a union of two open intervals, is open in $\mathbb{R}$.
(3) Every finite subset of a cofinite space is closed. Indeed let X be a cofinite space and $A \subset X$ be finite. Then $X \setminus (X \setminus A) = A$, a finite set. Hence $X \setminus A$ is open. So, A is closed.
(4) Every countable subset of a cocountable space is closed. This can be seen in the same way as in the cofinite space.
(5) $\mathbb{Q}$ is neither an open nor a closed subset of $\mathbb{R}$ with the usual topology. Indeed, if Q is open then it should contain some open interval but it does not. If Q is closed, then its complement should contain some open interval but it does not contain any.
(6) $\mathbb{Z}$ is a closed subset of $\mathbb{R}$ with the usual topology. Indeed $\mathbb{R} \setminus \mathbb{Z} = \cup_{k \in \mathbb{Z}} (k, k+1)$, a union of open sets, and hence, open. That is, $\mathbb{Z}$ is closed.
(7) $K = \{\frac{1}{n} : n \in \mathbb{N}\}$ is a closed subset of $\mathbb{R}_K$ because $\mathbb{R} \setminus K$ is closed by the Definition of $\mathscr{T}_K$.

(8) $[0,1)$ is open as well as closed subset of $\mathbb{R}_\ell$. Indeed by the definition of the lower limit topology, $[0,1)$ is open. Also $\mathbb{R} \setminus [0,1) = \cup_{n\in\mathbb{Z}\setminus\{0\}}[n, n+1)$

The family of closed sets in a topological space has the following properties, which are dual to those of open sets.

13.3. Theorem. *Let X be a topological space.*
(1) X and $\emptyset$ are closed.
(2) Arbitrary intersection of closed sets is closed.
(3) Finite union of closed sets is closed.

PROOF. ☞(1) Since $\emptyset$ and X are open, by the definition of a closed set, their complements X and $\emptyset$ are closed.

☞(2) Let $\{F_i : i \in I\}$ be a family of closed sets in X. Then $X \setminus F_i$ is open for every i. But then $X \setminus \cap_{i\in I} F_i = \cup_{i\in I}(X \setminus F_i)$ is open in X. Thus $\cap_{i\in I} F_i$ is closed in X.

☞(3) Let $\{F_i : 1 \le i \le n\}$ be a family of closed sets in X. Then $X \setminus F_i$ is open for every $1 \le i \le n$. So, $X \setminus \cup_{i=1}^n F_i = \cap_{i=1}^n (X \setminus F_i)$ is open in X. So, $\cup_{i=1}^n F_i$ is closed in X. □

13.4. Example. A finite set is closed in $\mathbb{R}$.

SOLUTION. Note that by the Example 13.2(2), every singleton is closed in $\mathbb{R}$ and a finite set, being a finite union of such closed sets, is closed by the Theorem 13.3(3). We still give a direct proof to instate the clarity of the subject in reader's mind.

Consider a finite subset $A = \{a_1, a_2, a_3, \ldots, a_n\}$ of $\mathbb{R}$. Without loss of generality, we assume that $a_1 < a_2 < a_3 < \cdots < a_n$ so that

$$\mathbb{R} \setminus A = (-\infty, a_1) \cup (a_2, a_3) \cup \cdots \cup (a_{n-1}, a_n) \cup (a_n, \infty),$$

which, being a union of open sets, is an open set. Hence A is closed in $\mathbb{R}$. □

The following converse to the Theorem 13.3 characterizes a topology and establishes the unique bonding between a topological space and the family of it's closed sets.

13.5. Theorem. *Let X be a nonempty set and $\mathscr{F} \subset \mathscr{P}(X)$ satisfy the following.*
(1) $X, \emptyset \in \mathscr{F}$.
(2) $\mathscr{F}$ is closed under arbitrary intersection.
(3) $\mathscr{F}$ is closed under finite union.
Define $\mathscr{T} = \{G \subset X : X \setminus G \in \mathscr{F}\}$. Then $\mathscr{T}$ is a topology on X and $\mathscr{F}$ is the family of all $\mathscr{T}$-closed subsets of X.

PROOF. $X, \emptyset \in \mathscr{F} \Rightarrow \emptyset, X \in \mathscr{T}$. Also for $\{G_i : i \in I\} \subset \mathscr{T}$, $\{X \setminus G_i : i \in I\} \subset \mathscr{F}$. Hence $X \setminus \cup G_i = \cap(X \setminus G_i) \in \mathscr{F}$ and so, $\cup G_i \in \mathscr{T}$. Also, for $G, H \in \mathscr{T}$,

$X \setminus G, X \setminus H \in \mathscr{F}$, and hence, $X \setminus (G \cap H) = (X \setminus G) \cup (X \setminus H) \in \mathscr{F}$. Thus $G \cap H \in \mathscr{T}$.

Finally, $F \subset X$ is $\mathscr{T}$-closed if and only if $X \setminus F \in \mathscr{T}$ if and only if $F \in \mathscr{F}$, which completes the proof. □

In passing we touch upon the concept of *door spaces*.

13.6. Definition. A topological space $(X, \mathscr{T})$ is said to be a *door space* if every subset of X is either $\mathscr{T}$-open or $\mathscr{T}$-closed.

13.7. Examples.

(1) An indiscrete space X is a door space if and only if X is singleton.
(2) A discrete space is always a door space because all its subsets are open.
(3) A VIP topology on a set X (cf. Example 7.7) is a door space. Indeed let X be a set and $a \in X$. Consider $\mathscr{T}_a = \{G \subset X : G = \emptyset \text{ or } a \in G\}$. Then for every proper $A \subset X$, either $a \in A$, in which case A is $\mathscr{T}$-open or $a \notin A$, in which case, A is closed.
(4) The same argument as the above shows that the excluded point topological space is also a door space.

13.8. Definition. A subset of a topological space is called a *clopen* set if it is open as well as closed. A topological space is said to be *0-dimensional* if it has a base consisting of clopen sets.

13.9. Proposition. *Let $(X, \mathscr{T})$ be a topological space and $A \subset Y$. Then A is $\mathscr{T}_Y$-closed if and only if there is a $\mathscr{T}$-closed set F such that $A = F \cap Y$.*

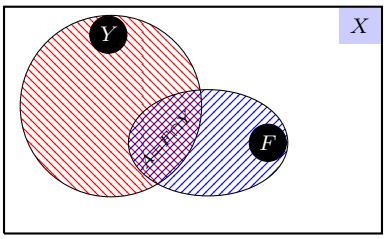


FIGURE 13. Closed in Y = (Closed in X) $\cap\, Y$

PROOF. Let $A \subset Y$ be $\mathscr{T}_Y$-closed. So that $Y \setminus A \in \mathscr{T}_Y$. Hence there is $G \in \mathscr{T}$ such that $Y \setminus A = G \cap Y$. Then $F = X \setminus G$ is $\mathscr{T}$-closed subset and

$$A = Y \setminus (Y \setminus A) = Y \setminus (G \cap Y) = (X \setminus G) \cap Y = F \cap Y.$$

Conversely, suppose $A = F \cap Y$ for some $\mathscr{T}$-closed subset F of X. But then $G = X \setminus F$ is $\mathscr{T}$-open and

$$A = F \cap Y = (X \setminus G) \cap Y = Y \setminus (G \cap Y).$$

Since $G \cap Y \in \mathscr{T}_Y$, A is $\mathscr{T}_Y$-closed. □

Recall the Proposition 12.6. A similar situation arises for closed sets too. In general, a closed subset of a subspace need not be closed in the main topological space. Indeed, let $X = \mathbb{R}$ and $Y = (0, 1)$. Then Y is closed in Y but Y is not closed in X. This may be true even for proper subsets too. Observe that $[0.1, 1)$ is closed in Y but not in X. The following asserts that *closed in closed is closed*; that is, open subset of an open subspace is an open subset.

13.10. Proposition. *Let $(X, \mathscr{T})$ be a topological space and Y be a closed subset of X. Then $A \subset Y$ is closed in X if and only if it is closed in Y.*

Proof. Let A be closed in X. Then $A = A \cap Y$ is closed in Y. Conversely, suppose A is closed in Y. So, by the Proposition 13.9, there is a $\mathscr{T}$-closed subset F of X such that $A = F \cap Y$. Since F and Y are $\mathscr{T}$-closed, so is A. □

Problems

98. Determine which of the following sets are either open or closed in the respective topologies. Which of them are the only proper closed sets of respective spaces?

(1) Finite subsets of a cofinite topological space.
(2) Countable subsets of a cofinite topological space.
(3) Countable subsets of a cocountable topological space.
(4) $(0, 1)$ in the discrete topology on $\mathbb{R}$.
(5) Finite subsets of $\mathbb{R}_\ell$.
(6) $(\sqrt{2}, \sqrt{3}) \cap \mathbb{Q}$ in $\mathbb{Q}$.
(7) $(2, 3) \cap \mathbb{Q}$ in $\mathbb{Q}$.
(8) Any subset of $\mathbb{Z}$ with the subspace topology.
(9) $\{\frac{1}{n} : n \in \mathbb{N}\}$ in the upper limit topology on $\mathbb{R}$.
(10) $\{\frac{1}{n} : n \in \mathbb{N}\}$ in the lower limit topology on $\mathbb{R}$.
(11) (a) $\mathbb{Z}$, (b) $\mathbb{N}$, (c) $\mathbb{Q}$ and (d) $[0, 1)$ in $\mathbb{R}$ with the usual topology.
(l) (a) $\mathbb{Z}$, (b) $\mathbb{N}$ (c) $\mathbb{Q}$, (d) $[0, 1)$, in $\mathbb{R}$ with the lower limit topology.

99. Show that in the following topological spaces, a subset is closed if and only if it is open. (1) A discrete space (2) An indiscrete space. (3) $X = \{1, 2, 3, 4\}$, $\mathscr{T} = \{\emptyset, X, \{1, 2\}, \{3, 4\}\}$.

100. Let X be a set and $a \in X$. Consider the VIP topology $\mathscr{T}_a = \{G \subset X : G = \emptyset$ or $a \in G\}$ on a set X. Show that every nonempty proper subset of X is either $\mathscr{T}$-open or $\mathscr{T}$-closed but not both.

101. Give examples from each of the following topological spaces to prove that arbitrary union of closed sets need not be closed. A set X with the (1) cofinite topology (2) cocountable topology (3) usual topology (4) lower limit topology.

102. Give three distinct topologies on $\mathbb{R}$ in which $\{0\}$ is not closed.

14. Closure, Interior and Limit Points

14.1. Definition. Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. We define *interior of A (with respect to X)* to be the set

$$\cup\{G \subset X : G \text{ is open in } X \text{ and } G \subset A\}.$$

It is denoted by A°, $\text{Int}(A)$ or $\text{Int}_X(A)$. If $x \in A^\circ$, then x is called an *interior point of A*. We also define *closure of A (with respect to X)* to be the set

$$\cap\{F \subset X : F \text{ is closed in } X \text{ and } A \subset F\}.$$

It is denoted by $\overline{A}$ or $\text{cl}(A)$ or $\text{cl}_X(A)$.

14.2. Remark. Based on the definition, we have the following observations.

(1) $x \in A^\circ$ if and only if A is a *neighbourhood of x*.
(2) A° is a union of (some) subsets of A. So, $A^\circ \subset A$.
(3) $\overline{A}$ is an intersection of (some) supersets of A. So, $A \subset \overline{A}$.
(4) Combining the above two, we have $A^\circ \subset A \subset \overline{A}$.
(5) A° is a union of (some) open subsets of X. So, A° is open in X.
(6) $\overline{A}$ is an intersection of (some) closed subsets of X. So, $\overline{A}$ is closed in X.
(7) A° is the largest open set in X contained in A. That is, if B is a open subset of X such that $B \subset A$, then $B \subset A^\circ$.
(8) $\overline{A}$ is the smallest closed set in X containing A. That is, if B is a closed subset of X such that $A \subset B$, then $\overline{A} \subset B$.
(9) It is not clear that $A \subset B \Rightarrow \overline{A} \subset \overline{B}$.
(10) From the context it will be clear with respect to which space, we are taking interior and closure of a set. Consequently, we shall drop the phrase *with respect to X* unless there is a need.
(11) Some authors define a neighbourhood to be open. In our case, a neighbourhood need not be open. For example, $[0, 5]$ is a neighbourhood of 1 in our framework, but it is not, according to the framework of some books.

14.3. Lemma. *Let $(X, \mathscr{T})$ be a topological space and $A \subset X$.*
(1) A is closed if and only if $A = \overline{A}$.
(2) A is open if and only if $A = A^\circ$.

Proof. ☞(1) Since $\overline{A}$ is closed, $A = \overline{A} \Rightarrow A$ is closed.

Conversely, suppose that A is closed. So, $A \in \{F \subset X : F \text{ is closed in } X \text{ and } A \subset F\}$. Hence $\cap\{F \subset X : F \text{ is closed in } X \text{ and } A \subset F\} \subset A$. That is, $\overline{A} \subset A$. Of course $A \subset \overline{A}$ is always true. Hence $A = \overline{A}$.

☞(2) Since A° is open, $A = A^\circ \Rightarrow A$ is open.

Conversely, suppose that A is open. So, $A \in \{G \subset X : G \text{ is open in } X \text{ and } G \subset A\}$. Hence $A \subset \cup\{G \subset X : G \text{ is open in } X \text{ and } G \subset A\}$. That is, $A \subset A^\circ$. Of course $A^\circ \subset A$ is always true. Hence $A = A^\circ$. □

Now we give Kuratowski closure axioms. First we note the following properties of the closure function.

14.4. Theorem. *Let $(X, \mathscr{T})$ be a topological space. The closure function $A \in \mathscr{P}(X) \mapsto \overline{A} \in \mathscr{P}(X)$ satisfies the following properties.*

(1) $A \subset \overline{A}$.
(2) $\overline{\overline{A}} = \overline{A}$.
(3) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(4) $\overline{\emptyset} = \emptyset$.
(5) A *is closed if and only if* $A = \overline{A}$.

PROOF. ☞(1) is already noted in the Remark 14.2 (3)

☞(2) follows from the Remark 14.2 (6) and the Lemma 14.3 (1).

☞(3) $A \subset A \cup B$ and $B \subset A \cup B$ so that $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. On the other hand, $A \cup B \subset \overline{A} \cup \overline{B}$. Hence $\overline{A \cup B} \subset \overline{\overline{A} \cup \overline{B}} = \overline{A} \cup \overline{B}$ because $\overline{A} \cup \overline{B}$ is closed.

☞(4) $\emptyset$ is closed, and so, $\overline{\emptyset} = \emptyset$.

☞(5) is nothing but the restatement of the Lemma 14.3 (1). □

The following is converse to the Theorem 14.4 and characterizes the topology in terms of the closure operation.

14.5. Theorem. *Let X be a set. Let $\chi : \mathscr{P}(X) \to \mathscr{P}(X)$ satisfy the following.*

(1) $A \subset \chi(A)$.
(2) $\chi(\chi(A)) = \chi(A)$.
(3) $\chi(A \cup B) = \chi(A) \cup \chi(B)$.
(4) $\chi(\emptyset) = \emptyset$.

Define the closed sets using the following.

(5) A *is closed if and only if* $A = \chi(A)$.

Let $\mathscr{F} = \{A \subset X : A = \chi(A)\}$ and $\mathscr{T} = \{G \subset X : X \setminus G \in \mathscr{F}\}$. Then $\mathscr{T}$ is a topology on X and for every $A \subset X$, $\overline{A} = \chi(A)$.

PROOF. First note that by (4), $\emptyset \in \mathscr{F}$. Also from (1), $X \subset \chi(X) \subset X$, and hence, $X = \chi(X)$. Thus $X \in \mathscr{F}$.

For $A \subset B$, using (3), $\chi(B) = \chi(A \cup (B \setminus A)) = \chi(A) \cup \chi(B \setminus A)$, which, in turn, gives $\chi(A) \subset \chi(B)$. That is, $A \subset B \Rightarrow \chi(A) \subset \chi(B)$.

Now let $\{E_i : i \in I\} \subset \mathscr{F}$. Let $j \in I$. Note that for every $j \in I$, $\cap_{i \in I} E_i \subset E_j$. Hence $\chi(\cap_{i \in I} E_i) \subset \chi(E_j)$. Taking intersection over $j \in I$, and observing that the left hand side of the latest expression, is independent of j, we get

$$\chi(\cap_{i \in I} E_i) \subset \cap_{j \in I} \chi(E_j) = \cap_{j \in I} E_j = \cap_{i \in I} E_i \subset \chi(\cap_{i \in I} E_i).$$

Thus $\cap \chi(E_i) \in \mathscr{F}$. Let $E_1, E_2, \ldots, E_n \in \mathscr{F}$. Then

$$\begin{aligned} \chi(E_1 \cup E_2 \cup \cdots \cup E_n) &= \chi(E_1) \cup \chi(E_2 \cup \cdots \cup E_n) \\ &= \chi(E_1) \cup \chi(E_2) \cup \chi(E_3 \cup \cdots \cup E_n) \\ &= \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \end{aligned}$$

$$= \chi(E_1) \cup \chi(E_2) \cup \chi(E_3) \cup \cdots \cup \chi(E_n).$$

Thus $\mathscr{F}$ satisfies all the properties stated in the Theorem 13.5. Hence $\mathscr{T}$ is a topology on X.

It remains to show that $\chi(A) = \overline{A}$ for every $A \subset X$. Let $A \subset X$. By (2), $\chi(\chi(A)) = \chi(A)$. Hence by (5), $\chi(A) \in \mathscr{F}$. Now let $B \in \mathscr{F}$ be such that $A \subset B$. Then $\chi(A) \subset \chi(B) = B$. Thus $\chi(A)$ is the smallest element of $\mathscr{F}$ that contains A. But $\overline{A}$ is the smallest element of $\mathscr{F}$ that contains A. Thus $\chi(A) = \overline{A}$. □

Similar to the above characterization of the topology by a closure operator, we also have the characterization of the topology by an interior operator also. We leave it in the Exercise 103 for the reader.

14.6. Remark. Applying Theorem 14.4 (3), conclude that the closure commutes with the finite union. That is, for subsets $A_1, A_2, A_3, \ldots, A_n$ of a topological space, $\cup_{i=1}^{n} \overline{A_i} = \overline{\cup_{i=1}^{n} A_i}$. However the following example shows that we cannot extend this for even countable union.

14.7. Example. By Example 13.2 (2), for every rational number r, $\{r\}$ is closed but as seen in Example 13.2 (5), $\mathbb{Q}$, the countable union of all this singletons, is not closed. Thus $\cup_{r\in\mathbb{Q}} \overline{\{r\}} \neq \overline{\cup_{r\in\mathbb{Q}} \{r\}}$.

However for a family with a certain characteristic, the closure respects arbitrary union.

14.8. Definition. Let X be a topological space. A family $\{A_i : i \in I\}$ of subsets of X is said to be locally finite, if for every $x \in X$, there is $U \in \mathscr{U}_x$ such that U meets only finitely many A_i. That is $\{i \in I : U \cap A_i \neq \emptyset\}$ is finite.

14.9. Theorem. *Let X be a topological space and $\{A_i : i \in I\}$ of subsets of X be locally finite. Then $\overline{\cup_{i\in I} A_i} = \cup_{i\in I} \overline{A_i}$.*

Proof. Note that $A_i \subset \cup_{i\in I} A_i$. So, by 14.2 (9), $\overline{A_i} \subset \overline{\cup_{i\in I} A_i}$. Consequently, $\cup_{i\in I} \overline{A_i} \subset \overline{\cup_{i\in I} A_i}$.

Conversely, let $x \in \overline{\cup_{i\in I} A_i}$. Choose $U \in \mathscr{U}_x$ such that $J = \{i \in I : U \cap A_i \neq \emptyset\}$ is finite. That is, for $i \in I \setminus J$, $U \cap A_i = \emptyset$. Hence $U \cap \cup_{i\in I\setminus J} A_i = \emptyset$. Thus $x \notin \overline{\cup_{i\in I\setminus J} A_i}$. Therefore, $x \in \overline{\cup_{i\in I} A_i} = \overline{\cup_{i\in J} A_i \cup \cup_{i\in I\setminus J} A_i} = \overline{\cup_{i\in J} A_i} \cup \overline{\cup_{i\in I\setminus J} A_i}$, Since $x \notin \overline{\cup_{i\in I\setminus J} A_i}$, we conclude that $x \in \overline{\cup_{i\in J} A_i} = \cup_{i\in J} \overline{A_i} \subset \cup_{i\in I} \overline{A_i}$. This completes the proof. □

14.10. Theorem. *Let $(X, \mathscr{T})$ be a topological space, $x \in X$, $\mathscr{B}$ be a base for $\mathscr{T}$, $\mathscr{B}_x$ be a local base at x and $A \subset X$. Then the following are equivalent.*

(1) $x \in \overline{A}$.
(2) $B \in \mathscr{B}_x \Rightarrow B \cap A \neq \emptyset$.
(3) $U \in \mathscr{U}_x \Rightarrow U \cap A \neq \emptyset$. That is, every neighbourhood of x also contains a point of A.
(4) Every open neighbourhood of x also contains a point of A.
(5) Every basic neighbourhood of x that is, a member of $\mathscr{B}$ containing x, also contains a point of A.

PROOF. ☞(1) ⇒ (2) Let $B \in \mathscr{B}_x$ and $G = B^\circ$. Then $x \in G \subset B$. Suppose, if possible, that $G \cap A = \emptyset$. So, $A \subset X \setminus G$, That is, $X \setminus G$ is a closed set containing A. But then $\overline{A} \subset X \setminus G$, which gives $x \in \overline{A} \subset X \setminus G$, a contradiction to $x \in G$. Hence $B \cap A \neq \emptyset$.

☞(2) ⇒ (3) follows from the fact that $\mathscr{U}_x$ is a local base at x.

☞(3) ⇒ (4) follows from the fact that every open neighbourhood is a neighbourhood.

☞(4) ⇒ (5) follows from the fact that every basic neighbourhood is an open neighbourhood.

☞(5) ⇒ (1) Suppose $x \notin \overline{A}$. Let $G = X \setminus \overline{A}$. Since $\overline{A}$ is closed, G is an open neighbourhood of x. Choose $B \in \mathscr{B}$ such that $x \in B \subset G$. But then by hypothesis, B contains a point of A. Hence, $\emptyset \neq B \cap A \subset G \cap \overline{A} = (X \setminus \overline{A}) \cap \overline{A} = \emptyset$, a contradiction. So, $x \in \overline{A}$. □

14.11. Remark. In order to say that "G contains a point of A", we also say that "G intersects A" which also means $G \cap A \neq \emptyset$.

14.12. Definition. Let $(X, \mathscr{T})$ be a topological space, $A \subset X$ and $x \in X$. We say that x is a *limit point* of A if every neighbourhood of x contains a point of A other than x. The set of all limit points of A is called the *derived set* of A and is denoted by A'.[4] A limit point is also called a *cluster point* or an *accumulation point*. The derived set of A is also denoted by $D(A)$ or $D_X(A)$.

14.13. Theorem. *Let $(X, \mathscr{T})$ be a topological space, $A \subset X$ and $x \in X$. The following are equivalent.*

(1) $x \in A'$.
(2) $x \in \overline{A \setminus \{x\}}$.

PROOF. ☞(1) ⇒ (2) Let $x \in A'$ and U be a neighbourhood of x. Then $\emptyset \neq A \cap U \setminus \{x\} = (A \setminus \{x\}) \cap U$. So, $x \in \overline{A \setminus \{x\}}$.

[4] Read A dash or A prime.

☞(2) ⇒ (1) Let $x \in \overline{A \setminus \{x\}}$ and $U \in \mathscr{U}_x$. Then $A \cap U \setminus \{x\} \neq \emptyset$. That is, U contains a point of A other than x. Thus $x \in A'$. □

Note

We write $A \cap U \setminus \{x\}$ without any bracketing because

$$(A \cap U) \setminus \{x\} = A \cap (U \setminus \{x\}) = (A \setminus \{x\}) \cap U.$$

However we cannot write $A \setminus \{x\} \cap U$ because the associativity of $\{x\}$ with $\setminus$ and $\cap$ give different results.

14.14. Theorem. *Let X be a topological space and $A \subset X$. Then $\overline{A} = A \cup A'$.*

Proof. Clearly, $A \subset \overline{A}$. Also, $A' = \overline{A \setminus \{x\}} \subset \overline{A}$. Thus $A \cup A' \overline{A}$. Conversely, suppose $x \in \overline{A}$. If $x \in A$, then we are done as $x \in A \subset A \cup A'$. So suppose $x \notin A$. Let U be a neighbourhood of x. Since $x \in \overline{A}$, U contains a point of A, say $y \in U \cap A$. Since $y \in A$ and $x \notin A$, $x \neq y$. Thus $y \in A \setminus \{x\}$. Thus U contains a point of A other than x. Thus $x \in A' \subset A \cup A'$. Thus $\overline{A} \subset A \cup A'$. □

14.15. Corollary. *Let X be a topological space and $A \subset X$. Then A is closed if and only if $A' \subset A$.*

Proof. If A is closed, then $A' \subset A \cup A' = \overline{A} = A$. On the other hand, if $A' \subset A$, then $\overline{A} = A \cup A' \subset A \cup A = A$. □

The closed sets are precisely those sets A for which $\overline{A} = A$, equivalently, $A' \subset A$. The sets A for which $A' = A$ are also interesting.

14.16. Definition. Let X be a topological space. $A \subset X$ is said to be *perfect* if $A' = A$.

14.17. Remark. Since $A = A'$ for a perfect set A, it is closed. However, the closed set $\{1\}$ is not perfect because 1 is not it's limit point.

14.18. Example.

(1) $[0, 1]$ is perfect.
(2) $\mathbb{R}$ is perfect.
(3) No subset of a discrete space is perfect.
(4) No singleton can be perfect.
(5) $\emptyset$ is perfect.

14.19. Definition. Let X be a topological space and $A \subset X$. We define the *boundary of A* to be the set $\overline{A} \cap \overline{(X \setminus A)}$. It is denoted by $\partial(A)$ or $\mathrm{bd}(A)$. $x \in X$ is said to be a *boundary point of A* if $x \in \mathrm{bd}(A)$.

It is clear from the definition of the boundary of a set that

(1) $\mathrm{bd}(A) = \mathrm{bd}(X \setminus A)$ and
(2) $\mathrm{bd}(A)$ is closed.

14.20. Proposition. *Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. Then A is closed if and only if $\mathrm{bd}(A) \subset A$.*

Proof. If A is closed, then $\mathrm{bd}(A) = \overline{A} \cap \overline{(X \setminus A)} \subset \overline{A} = A$. Conversely, assume that $\mathrm{bd}(A) \subset A$. Suppose, if possible, that $x \in \overline{A}, x \notin A$. Hence $x \in \overline{A} \cap (X \setminus A) \subset \overline{A} \cap \overline{(X \setminus A)} = \mathrm{bd}(A) \subset A$, a contradiction. Hence $x \in A$. Thus $\overline{A} \subset A$. □

14.21. Corollary. *Let $(X, \mathscr{T})$ be a topological space and $A \subset X$. Then A is open if and only if $\mathrm{bd}(A) \cap A = \emptyset$.*

Proof. A is open $\Leftrightarrow X \setminus A$ is closed $\Leftrightarrow \mathrm{bd}(A) = \mathrm{bd}(X \setminus A) \subset X \setminus A \Leftrightarrow \mathrm{bd}(A) \cap A = \emptyset$. □

14.22. Proposition. *Let X be a topological space and $A, B \subset X$. Then*

(1) $\mathrm{Int}(A) = A \setminus \mathrm{bd}(A)$.
(2) $\overline{A} = A \cup \mathrm{bd}(A)$.

Proof. ☞ (1) Let $\mathrm{Int}(A)$ is an open set disjoint from $X \setminus A$. So, $\mathrm{Int}(A) \cap \mathrm{bd}(A) \subset \mathrm{Int}(A) \cap \overline{X \setminus A} = \emptyset$. Also, $\mathrm{Int}(A) \subset A$. Consequently, $\mathrm{Int}(A) \subset A \setminus \mathrm{bd}(A)$.

☞ (2) Let $x \in A \setminus \mathrm{bd}(A)$. Since $x \notin \mathrm{bd}(A)$, there is a neighbourhood $U \in \mathscr{U}_x$ such that $U \subset A$ or $U \subset X \setminus A$. But $x \in A$, thus $U \subset A$. Hence, $x \in \mathrm{Int}(A)$. This completes the proof. □

14.23. Proposition. *Let $(X, \mathscr{T})$ be a topological space and Y be a subspace of X. For $A \subset Y$,*

(1) $D_Y(A) = D_X(A) \cap Y$.
(2) $\mathrm{cl}_Y(A) = \mathrm{cl}_X(A) \cap Y$.

Proof. ☞(1) Let $x \in D_Y(A)$ and $G \in \mathscr{U}_x$. Hence $G \cap Y$ is a neighbourhood of x in Y. So, $G \cap Y$, and hence, G contains a point of $A \setminus \{x\}$. Thus $x \in D_X(A)$.

Conversely, for $x \in D_X(A)$, let G be a neighbourhood of x in Y. So, there is a neighbourhood H of x in X such that $G = H \cap Y$. But then H contains a point of $A \setminus \{x\}$. Since $A \subset Y$, $\emptyset \neq (A \setminus \{x\}) \cap H = ((A \setminus \{x\}) \cap Y) \cap H = (A \setminus \{x\}) \cap (H \cap Y) = (A \setminus \{x\}) \cap G$. Hence $x \in D_Y(A)$.

☞(2) We prove this in two different ways. One is from (1).

$$\mathrm{cl}_Y(A) = A \cup A'_Y = A \cup (A'_X \cap Y) = (A \cup A'_X) \cap (A \cup Y) = \mathrm{cl}_X(A) \cap Y.$$

For the second proof clearly, $\mathrm{cl}_X(A) \cap Y$ is a closed subset of Y containing A. So,

$$\mathrm{cl}_Y(A) \subset \mathrm{cl}_X(A) \cap Y. \tag{14.23.1}$$

Now let $E \subset Y$ be a closed subset of Y containing A. Hence $E = F \cap Y$ for some closed set F of Y. But then $A \subset E \subset F$. That is, F is a closed subset of X containing A. Hence $\mathrm{cl}_X(A) \subset F$. That means $\mathrm{cl}_X(A) \cap Y \subset F \cap Y = E$. Thus every closed subset of Y containing A also contains $\mathrm{cl}_X(A) \cap Y$. But $\mathrm{cl}_Y(A)$ is a closed subset of Y containing A. Hence $\mathrm{cl}_Y(A)$ contains $\mathrm{cl}_X(A) \cap Y$. Thus $\mathrm{cl}_X(A) \cap Y \subset \mathrm{cl}_Y(A)$. This proves that $\mathrm{cl}_Y(A) = \mathrm{cl}_X(A) \cap Y$. □

The idea of closed set is dual to that of open sets. The idea of closure is dual to that of interior. However, the results similar to the above does not hold for the interior.

14.24. Example. Let $X = \mathbb{R}, Y = A = [0, 1]$. Then

$$\mathrm{Int}_X(A) \cap Y = (0, 1) \cap [0, 1] \neq [0, 1] = \mathrm{Int}_Y(A).$$

14.25. Definition. Let X be a topological space. $A \subset X$ is said to be

(1) *dense in X* if $\overline{A} = X$.
(2) *nowhere dense in X* if $(\overline{A})^\circ = \emptyset$. Here closure and interior are with respect to X.
(3) *of first category in X* if it is a countable union of nowhere dense sets. That is, if there exist nowhere dense subsets $A_1, A_2, \ldots$ of X such that $A = \cup_{n=1}^{\infty} A_n$.
(4) *second category in X* if it is not of first category in X.

14.26. Example.

(1) $\mathbb{Z}$ is
  (a) nowhere dense in $\mathbb{R}$ because $\overline{\mathbb{Z}} = \mathbb{Z}$, and hence, $(\overline{\mathbb{Z}})^\circ = \emptyset$ where the operations are with respect to $\mathbb{R}$. In fact, every set is dense in itself.
  (b) not nowhere dense in $\mathbb{Z}$ because $\overline{\mathbb{Z}} = \mathbb{Z}$, where the closure is taken with respect to $\mathbb{Z}$.
  (c) of first category in $\mathbb{R}$ because it is nowhere dense in $\mathbb{R}$.
  (d) of second category in $\mathbb{Z}$ for any subset $A \subset \mathbb{Z}$, $A = \overline{A} = A^\circ$, where the closure and interior are taken with respect to $\mathbb{Z}$.
(2) $\mathbb{Q}$ is
  (a) dense in $\mathbb{R}$ because $\overline{\mathbb{Q}} = \mathbb{R}$, where the closure is taken with respect to $\mathbb{R}$.
  (b) not nowhere dense in $\mathbb{Q}$ because $\overline{\mathbb{Q}} = \mathbb{Q}$, where the closure is taken with respect to $\mathbb{Q}$.
  (c) of first category in $\mathbb{R}$ because $\mathbb{Q}$ is a countable union of singletons and every singleton is closed and has empty interior in $\mathbb{R}$.
  (d) of first category in $\mathbb{Q}$ because $\mathbb{Q}$ is a countable union of singletons and every singleton is closed and has empty interior in $\mathbb{Q}$.

Problems

103. Dual to the Kuratowski closure axioms, obtain the results for the interior operator on a topological space and also characterize the topology by means of an interior operator given on a set.

104. Let $X = \{1, 2, 3, 4\}$, $\mathscr{T} = \{\emptyset, X, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$. Find the interior and closure of $A = \{1, 4\}$ and $B = \{1, 2\}$.

105. Find the closure and interior of $[0, 1], (0, 1), [0, 1), (0, 1], \mathbb{Q}, \mathbb{Z}, \{\frac{1}{n} : n \in \mathbb{N}\}$ in $\mathbb{R}$ with respect to each of the topologies: (1) cofinite, (2) cocountable, (3) usual and (4) lower limit.

106. Let X be a topological space and $A, B \subset X$. Answer the following.

(1) Prove that $(A^\circ)^\circ = A^\circ$.
(2) Prove that $\overline{\overline{A}} = \overline{A}$.
(3) If $A \subset B$, then show that $A^\circ \subset B^\circ$ and $\overline{A} \subset \overline{B}$. Does the converse hold for any one of them?
(4) Show that $(A \cap B)^\circ = A^\circ \cap B^\circ$ and $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(5) Prove that $X \setminus \overline{A} = (X \setminus A)^\circ$ and $X \setminus A^\circ = \overline{X \setminus A}$.
(6) Show that $A^\circ \cup B^\circ \subset (A \cup B)^\circ$ and $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. Give examples to show that equality may not hold in each of them.
(7) Prove or disprove: $\overline{A \setminus B} = \overline{A} \setminus \overline{B}$.
(8) Prove or disprove: $(A \setminus B)^\circ = A^\circ \setminus B^\circ$.

107. Find the limit points of $(0, 1)$, $[0, 1]$, $[0, 1)$, $\mathbb{Q}$, $\mathbb{N}$ in $\mathbb{R}$ with the following topologies on $\mathbb{R}$. (1) usual (2) lower limit (3) cocountable (4) cofinite (5) $\mathscr{T} = \{G \subset X : G = \emptyset, G = \mathbb{R} \text{ or } G = (a, \infty), a \in \mathbb{R}\}$.

108. Find the limit points of $\{1, 2\}$ in $X = \{1, 2, 3, 4\}$ with the topology $\mathscr{T} = \{\emptyset, X, \{1\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$.

109. Give examples of a countable subset A of $\mathbb{R}$ for each of the following situations.

(1) $A' = \emptyset$.
(2) A' contains exactly n elements. $n \in \mathbb{N}$ but A is not closed.
(3) A' contains exactly n elements. $n \in \mathbb{N}$ but A is closed.
(4) A' is countably infinite and A is not closed.
(5) A' is countably infinite and A is closed.
(6) A' is uncountable.

110. Give examples of following subsets A of some topological space in which $A' \neq \emptyset$. (1) $A' \cap A = \emptyset$ (2) $A' \cap X \setminus A = \emptyset$ (3) $A' \cap A \neq \emptyset \neq A' \cap X \setminus A$.

111. Give an example of a set A such that $A \subsetneq \mathrm{bd}(A)$.

112. For a subset A of a topological space X, show that

(1) $\overline{A} = A \cup \mathrm{bd}(A)$.
(2) $A^\circ = A \setminus \mathrm{bd}(A)$.
(3) $X = A^\circ \cup \mathrm{bd}(A) \cup (X \setminus A)^\circ$.

113. Let X be a set with two topologies $\mathscr{T}_1 \subset \mathscr{T}_2$. For a subset A of X, let $A_i^\circ, \overline{A}_i, A_i', \mathrm{bd}_i(A)$ denote the interior, closure, derived set and boundary of A with respect to the $\mathscr{T}_i$, $i = 1, 2$. Show that (1) $A_1^\circ \subset A_2^\circ$, (2) $\overline{A}_2 \subset \overline{A}_1$, (3) $A_2' \subset A_1'$, and (4) $\mathrm{bd}\, A_2 \subset \mathrm{bd}\, A_1$. Give examples to show that the equality need not hold in any of the inclusions.

114. Show that a topological space is discrete if and only if it does not have a proper dense subset.

115. Let X be a topological space with local bases $\mathscr{B}_x$ at all points $x \in X$. Let $A \subset X$. Show that $x \in \mathrm{bd}(A)$ if and only if each $B \in \mathscr{B}_x$ intersects both A as well as $X \setminus A$.

15. Order Topology

Let $(X, \leq)$ be a linearly ordered set, $a, b, c, d \in X$ and $a < b$ and $c < d$. Then one can easily prove that $(a, b) \cap (c, d) = (e, f)$, where $e = \max\{a, c\}$ and $f = \min\{d, f\}$. Also, $(-\infty, b) \cap (a, \infty) = (a, b)$. This is precisely what is used to construct basis and subbasis for a topology on X.

15.1. Proposition. *Let X be a linearly ordered set.*

(1) $\mathscr{B}_1 = \{(-\infty, a) : a \in X\} \cup \{(a, \infty) : a \in X\} \cup \{(-\infty, \infty)\} \cup \{(a, b) : a < b, a, b \in X\}$ is a base for some topology on X.

(2) $\mathscr{S} = \{(-\infty, a) : a \in X\} \cup \{(a, \infty) : a \in X\} \cup \{(-\infty, \infty)\}$ is a subbase for the topology generated by $\mathscr{B}_1$.

(3) If X does not have the maximum element and minimum element, then $\mathscr{B}_2 = \{(a, b) : a < b, a, b \in X\}$ is a base for some topology on X.

(4) If X does not have the maximum element and minimum element, then $\mathscr{B}_1$ and $\mathscr{B}_2$ generate the same topology.

Proof. ☞ (1) Clearly, $X = (-\infty, \infty) \subset \cup\mathscr{B}_1$. On the other hand, let $B_1, B_2 \in \mathscr{B}_1$ and $x \in B_1 \cap B_2$. Then $B_1 \cap B_2 = (e, f) \in \mathscr{B}_1$ as shown in the various cases as follows.

$$
\begin{array}{llll}
B_1 \cap B_2 = (a, b) \cap (c, d) = (e, f) & \text{with } e = \max\{a, c\} & f = \min\{b, x\} \\
B_1 \cap B_2 = (a, b) \cap (c, \infty) = (e, f) & \text{with } e = \max\{a, c\} & f = b \\
B_1 \cap B_2 = (a, b) \cap (-\infty, d) = (e, f) & \text{with } e = a & f = \min\{b, d\} \\
B_1 \cap B_2 = (a, b) \cap (-\infty, \infty) = (e, f) & \text{with } e = a & f = b \\
B_1 \cap B_2 = (a, \infty) \cap (c, \infty) = (e, f) & \text{with } e = \max\{a, c\} & f = \infty \\
B_1 \cap B_2 = (a, \infty) \cap (-\infty, d) = (e, f) & \text{with } e = a & f = b \\
B_1 \cap B_2 = (a, \infty) \cap (-\infty, \infty) = (e, f) & \text{with } e = a & f = \infty \\
B_1 \cap B_2 = (-\infty, b) \cap (-\infty, d) = (e, f) & \text{with } e = a & f = \min\{b, d\}.
\end{array}
$$

This proves that $\mathscr{B}_1$ is a base.

☞ (2) The proof follows by noting that for $a < b$, $(a, b) = (-\infty, b) \cap (a, \infty)$.

☞ (3) Let $x \in X$. Since $\max X, \min X$ do not exist, there are $a, b \in X$ such that $x \in (a, \infty) \cap (-\infty, b) = (a, b)$. Also, for $(a, b), (c, d) \in \mathscr{B}_2$, with $x \in (a, b) \cap (c, d)$, we have $(a, b) \cap (c, d) = (e, f) \in \mathscr{B}$, where $e = \max\{a, c\}$ and $f = \min\{b, d\}$. Thus $\mathscr{B}_2$ is a base for some topology on X.

☞ (4) Let $\mathscr{T}_1, \mathscr{T}_2$ be the topologies generated by $\mathscr{B}_1, \mathscr{B}_2$ respectively. For every $b \in X$, choose c_b, d_b such that $c_b < b < d_b$. Then for every $a \in X$,

$$(a, \infty) = \cup_{b \geq a}(a, d_b)$$

$$(-\infty, a) = \cup_{b \le a}(c_b, a)$$
$$(-\infty, \infty) = (-\infty, d_a) \cup (c_a, \infty)$$

are $\mathscr{T}_2$-open. Thus $\mathscr{T}_1 \subset \mathscr{T}_2$. Since $\mathscr{B}_2 \subset \mathscr{B}_1$, the reversed inclusion is obvious. Thus $\mathscr{T}_1 = \mathscr{T}_2$. □

15.2. Remark.

(1) We need to include $(-\infty, \infty) \in \mathscr{B}_1$ only to take care of the case when $X = \{a\}$, a singleton. Note that in this case, $(a, \infty) = (-\infty, a) = \emptyset$ and (a, b) does not exist because there do not exist two points a, b with $a < b$. Thus $\mathscr{B}_2$, would be empty if we would not have included $(-\infty, \infty)$. However, if there are two distinct points $a < b$ in X, then $(a, \infty), (-\infty, b) \in \mathscr{B}_1$ and $(-\infty, b) \cup (a, \infty) = X$.

(2) Irrespective of X, the intersection of empty family of members of $\mathscr{S}$ gives $X \in \mathscr{B}_1$.

15.3. Definition. Let $(X, \le)$ be a linearly ordered set. Then the topology generated by $\mathscr{B}_1$ in the Proposition 15.1 is called the *order topology on X*. X with the order topology is called an *ordered topological space*.

15.4. Remark. The Example 8.9 defines the usual topology on $\mathbb{R}$ which is nothing but the order topology on $\mathbb{R}$.

15.5. Example. On a totally ordered set X, $\mathscr{T} = \{(a, \infty) : a \in X\} \cup \{(-\infty, \infty)\}$ is a topology, called the *right order topology* on X.

15.6. Example. On a totally ordered set X, $\mathscr{T} = \{(-\infty, a) : a \in X\} \cup \{(-\infty, \infty)\}$ is a topology, called the *left order topology* on X.

The subspace of an ordered topological space need not be an ordered topological space. To discuss this, we need one definition and one notation.

15.7. Definition. Let X be a linearly ordered set. $Y \subset X$ is said to be convex if $a, b \in Y, a < x < b \Rightarrow x \in Y$.

Let X be a linearly ordered set and $Y \subset X$. For $a, b \in Y$, $(a, b)_X = \{x \in X : a < x < b\}$ and $(a, b)_Y = \{x \in Y : a < x < b\} = (a, b)_X \cap Y$.

15.8. Example. Subspace of an ordered space need not be an ordered space.

Indeed consider $X = \mathbb{R}$ and $Y = (0, 1) \cup \{2\}$. Then $\{2\}$ is an open set in Y with the subspace topology inherited from $\mathbb{R}$. On the other hand, the open sets containing 2 in the ordered topology on Y are $(a, \infty) = (a, 1) \cup \{2\}$. Thus $\{2\}$ is not open in the order topology on Y.

However, we have the following.

15.9. Theorem. *Let X be an ordered space and $Y \subset X$. Then the order topology on Y is coarser than the subspace topology. If Y is convex, then both the topologies coincide.*

PROOF. A subbasic open set in Y with the order topology is of the form

$$(-\infty, a)_Y = \{y \in Y : y < a\} = \{y \in X : y < a\} \cap Y = (-\infty, a)_X \cap Y$$
$$(a, \infty)_Y = \{y \in Y : a < y\} = \{y \in X : a < y\} \cap Y = (a, \infty)_X \cap Y,$$

where $a \in Y$, both of which are open in the subspace topology. Thus, the order topology is weaker than the subspace topology. Note that the culprit cold be the sets of the form $(-\infty, a)_X \cap Y$ or $(a, \infty)_X \cap Y$, where $a \in X \setminus Y$. But if Y is convex, then this sets could be either $\emptyset$ or the whole of Y, which are open in the order topology on Y. □

It is not necessary for a subspace to be convex in order to have the coincidence of the subspace topology with the order topology. In fact, $\mathbb{Q}$ is not convex, but both the topologies coincide on $\mathbb{Q}$.

15.10. Theorem. *Let X be an ordered space and $a < b$ in X. Then*
(1) $[a, b]$ is closed.
(2) $\overline{[a, b)} = [a, b]$ or $[a, b)$.

PROOF. ☞(1) $X \setminus [a, b] = (-\infty, a) \cup (b, \infty)$, an open set. Thus $[a, b]$ is closed.

☞(2) If the immediate predecessor of b exists, say c, then $b \in (c, \infty)$, which is an open set not intersecting with $[a, b)$. Even the direct verification gives $[a, b) = [a, c]$, a closed set. If the immediate predecessor of b does not exist, for every open set G containing b, there is $c < b < d$ such that $(c, d) \subset G$. Clearly, $\emptyset \neq (c, b) \cap [a, b) \subset G \cap [a, b)$. Thus b is a limit point of $[a, b)$. Clearly, $[a, b]$ is a closed set and hence it is the smallest closed set containing $[a, b)$. Thus, in this case, $\overline{[a, b)} = [a, b]$. □

15.11. Example.

(1) The order topology on $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ are the usual topologies.
(2) The order topology on $(0, 1) \cup \{2\}$ is not the usual topology.
(3) The order topology on $[0, 1] \cup \{2\}$ is the usual topology.

CHAPTER 3

Continuous Functions

Continuous functions are known to every student. In fact, the concept of continuity is usually introduced in school mathematics. However, the scope of study of continuous function in school mathematics, Calculus as well as the classical Analysis course is limited to the continuity of real-valued or complex-valued functions defined on subsets of $\mathbb{R}$ or $\mathbb{C}$. Here we define continuity in the general setup of topological spaces. Later on we shall prove that in the classical case, the concept of this continuity coincides with the usual concept.

16. Continuous Functions

16.1. Definition. Let X and Y be topological spaces, $f\colon X \to Y$ be a function and $x \in X$. We say that *f is continuous at x* if for every $V \in \mathscr{U}_{f(x)}$, there exists $U \in \mathscr{U}_x$ such that $f(U) \subset V$. f is said to be *continuous on X* if f is continuous at each point of X. We also define

$$C(X,Y) = \{f\colon X \to Y : f \text{ is continuous on } X\}.$$

As we can see continuity of a function at a point is a local phenomenon.

16.2. Lemma. *Let X and Y be topological spaces and $f\colon X \to Y$ be a function. Then f is continuous at $x \in X$ if and only if*

$$\{f^{-1}(V) : V \in \mathscr{U}_{f(x)}\} \subset \mathscr{U}_x.$$

That is, inverse image of a neighbourhood of $f(x)$ is a neighbourhood of x.

Proof. Suppose f is continuous at $x \in X$. Let $V \in \mathscr{U}_{f(x)}$. So, by the definition of the continuity, there is $U \in \mathscr{U}_x$ such that $f(U) \subset V$. Hence $U \subset f^{-1}(f(U)) \subset f^{-1}(V)$. Since superset of a neighbourhood of x is again a neighbourhood of x, it follows that $f^{-1}(V) \in \mathscr{U}_x$. On the other hand, suppose $\{f^{-1}(V) : V \in \mathscr{U}_{f(x)}\} \subset \mathscr{U}_x$. Let $V \in \mathscr{U}_{f(x)}$. So, $f^{-1}(V) \in \mathscr{U}_x$. So that $x \in f^{-1}(V)^\circ \subset f^{-1}(V)$. Taking $U = f^{-1}(V)^\circ$, the proof follows. □

16.3. Theorem. *Let X and Y be topological spaces, $f\colon X \to Y$ be a function. The following are equivalent.*

(1) f is continuous on X.
(2) For every open set $V \subset Y$, $f^{-1}(V)$ is open in X.

PROOF. ☞(1) ⇒ (2) Let $V \subset Y$ be an open set. Let $a \in f^{-1}(V)$. Then there is an open set $U_a \subset X$ such that $a \in U_a$ and $f(U_a) \subset V$. Clearly, $U = \cup_{a\in f^{-1}(V)} U_a$ is open in X. For $x \in U$, clearly, $x \in U_x \subset f^{-1}(f(U_x)) \subset f^{-1}(V)$. Also for $x \in f^{-1}(V)$, $x \in U_x \subset U$. Thus $f^{-1}(V) = U$, which is open.

☞(2) ⇒ (1) Let $a \in X$ and $V \subset Y$ be open set such that $f(a) \in V$. Taking $U = f^{-1}(V)$, we get $a \in U$ and $f(U) \subset V$. This completes the proof. □

16.4. Theorem. *Let X, Y be topological spaces and $f\colon X \to Y$ be a function. Then f is continuous if and only if $f^{-1}(F)$ is closed for every is closed subset F of Y.*

PROOF. For a closed set F in Y and $U = X \setminus F$,

$$f^{-1}(U) = f^{-1}(Y \setminus (Y \setminus U)) = X \setminus f^{-1}(Y \setminus U) = X \setminus f^{-1}(F)$$

shows that $f^{-1}(U)$ is open if and only if $f^{-1}(F)$ is closed. This completes the proof. □

16.5. Theorem. *Let X, Y be topological spaces and $f\colon X \to Y$ be a function. Then f is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for every subset A of X.*

PROOF. Suppose f is continuous. Let $B = \overline{f(A)}$. Then $A \subset f^{-1}(f(A)) \subset f^{-1}(\overline{f(A)})$, a closed set in X. Thus $\overline{A} \subset f^{-1}(\overline{f(A)})$. Applying f on both the sides, we have $f(\overline{A}) \subset f(f^{-1}(\overline{f(A)})) \subset \overline{f(A)}$.

Conversely, let $B = \overline{B} \subset Y$ and $A = f^{-1}(B)$. For $x \in \overline{A}$, $f(x) \in f(\overline{A}) \subset \overline{f(A)} \subset \overline{B} = B$. So that $x \in f^{-1}(B) = A$. Thus $f^{-1}(B)$ is closed, implying the continuity of f. □

The following is usually not found in the modern textbooks. It is discussed in [**Eng89**, Proposition 1.4.1].

16.6. Corollary. *Let X, Y be topological spaces and $f\colon X \to Y$ be a function. Then f is continuous if and only if $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for every subset B of Y if and only if $f^{-1}(\mathrm{Int}(B)) \subset \mathrm{Int}(f^{-1}(B))$ for every subset B of Y.*

PROOF. Assume that f is continuous. Let $A = f^{-1}(B)$. Then by Theorem 16.5 $f(\overline{f^{-1}(B)}) = f(\overline{A}) \subset \overline{f(A)} = \overline{f(f^{-1}(B))} \subset \overline{B}$. Applying f^{-1} both the sides, we have $\overline{f^{-1}(B)} \subset f^{-1}(f(\overline{f^{-1}(B)})) \subset f^{-1}(B)$.

Now assume that for every $B \subset Y$, $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$. Then for $B \subset Y$, $\overline{f^{-1}(Y \setminus B)} \subset f^{-1}(\overline{Y \setminus B})$. Hence,

$$\begin{aligned} f^{-1}(\text{Int}(B)) &= f^{-1}(Y \setminus (\overline{Y \setminus B})) \\ &= X \setminus f^{-1}(\overline{Y \setminus B}) \\ &\subset X \setminus \overline{f^{-1}(Y \setminus B)} \\ &= X \setminus \overline{X \setminus f^{-1}(B)} \\ &= \text{Int}(f^{-1}(B)). \end{aligned}$$

Finally, assume that $f^{-1}(\text{Int}(B)) \subset \text{Int}(f^{-1}(B))$ for every subset B of Y. Clearly, for an open set $U \subset Y$, we have $U = \text{Int}(A)$. Consequently, $f^{-1}(U) = f^{-1}(\text{Int}(U)) \subset \text{Int}(f^{-1}(U)) \subset f^{-1}(U)$. Thus $f^{-1}(U)$ is open for every open subset U of Y. Thus f is continuous. □

16.7. Theorem. *Let $(X, \mathscr{T}), (Y, \mathscr{T}')$ be topological spaces, $\mathscr{S}$ be a subbase for $\mathscr{T}'$ generating the base $\mathscr{B}$ for $\mathscr{T}'$ and $f : X \to Y$ be a function. The following are equivalent.*

(1) f is continuous.
(2) $f^{-1}(S)$ is open in X for every $S \in \mathscr{S}$.
(3) $f^{-1}(B)$ is open in X for every $B \in \mathscr{B}$.

PROOF. ☞(1) ⇒ (2) follows from the fact that $\mathscr{S} \subset \mathscr{T}$.

☞(2) ⇒ (3) Let $B \in \mathscr{B}$. Then $B = \cap_{i=1}^{n} S_i$ with $S_i \in \mathscr{S}$, $(1 \leq i \leq n)$. But then $f^{-1}(B) = f^{-1}(\cap_{i=1}^{n}(S_i)) = \cap_{i=1}^{n} f^{-1}(S_i) \in \mathscr{T}$. This proves (3).

☞(3) ⇒ (1) Assume (3). Let $V \in \mathscr{T}'$. Then $V = \cup_{i \in I} B_i$ with $B_i \in \mathscr{B}$ for every $i \in I$. But then $f^{-1}(V) = f^{-1}(\cup_{i \in I} B_i) = \cup_{i \in I} f^{-1}(B_i) \in \mathscr{T}$ because $f^{-1}(B_i) \in \mathscr{T}$ for every $i \in I$. □

16.8. Proposition. *Composition of two continuous functions is continuous.*

PROOF. Let X, Y, Z be topological spaces, and $f : X \to Y$ and $g : Y \to Z$ be two continuous functions. Let $V \subset Z$ be open. By the continuity of g, $g^{-1}(V)$ is open in Y. But then by the continuity of f, $(g \circ f)^{-1} = f^{-1}(g^{-1}(V))$ is open in X. Hence $g \circ f$ is continuous. □

16.9. Definition. Let X be a topological space and $f : X \to \mathbb{R}$. $f : X \to \mathbb{R}$ is said to be *upper semicontinuous* if f is continuous with respect to the right order topology on $\mathbb{R}$. That is $\{x \in X : f(x) > a\}$ is open in X for every $a \in \mathbb{R}$. $f : X \to \mathbb{R}$ is said to be *lower semicontinuous* if f is continuous with respect to the left order topology on $\mathbb{R}$. That is $\{x \in X : f(x) < a\}$ is open in X for every $a \in \mathbb{R}$.

Problems

116. Show that a one-one function from a cofinite topological space to itself is continuous.

117. Show that a one-one function from a cocountable topological space to itself is continuous.

118. Give an example of a bijection $f\colon \mathbb{R} \to \mathbb{R}$ which is not continuous.

119. Show that the characteristic function $\chi_{[0,1)}\colon \mathbb{R}_\ell \to \mathbb{R}_\ell$ is continuous.

120. Show that the characteristic function $\chi_{(0,1]}\colon \mathbb{R}_\ell \to \mathbb{R}_\ell$ is discontinuous.

121. For a topological space X and a function $f\colon X \to \mathbb{R}$, show that f is continuous with respect to the usual topology if and only if f is upper as well as lower semicontinuous.

122. For a topological space X and a function $f\colon X \to \mathbb{R}$, show that f is upper semicontinuous if and only if $-f$ is lower semicontinuous.

123. Let X be a topological space and $f, g\colon X \to \mathbb{R}$ be functions. If f, g are upper semicontinuous, then show that so are $\min\{f, g\}$, $\max\{f, g\}$, $f + g$. Further, if $f(X), g(X) \subset [0, \infty)$, then show that fg is also upper semicontinuous.

124. Let $f\colon (0, 1) \to \mathbb{R}$ be a continuous function. Suppose there exist real numbers l, m such that for every $\varepsilon > 0$, there is $\delta > 0$ such that $x < \delta \Rightarrow |f(x) - l| < \varepsilon$ and $1 - x < \delta \Rightarrow |f(x) - m| < \varepsilon$. Define $f(0) = l$ and $f(1) = m$. Show that the new definition of f makes it continuous at on $[0, 1]$.

125. Determine the continuity of each $f\colon (\mathbb{R}, \mathscr{T}_1) \to (\mathbb{R}, \mathscr{T}_2)$ defined by the following formula in every pair of topologies $\mathscr{T}_1$ and $\mathscr{T}_2$.

☞ Formula for f

(1) x (2) $-x$ (3) $\sin x$ (4) $\cos x$ (5) x^2 (6) x^3.

☞ Pairs of topologies

No.	$\mathscr{T}_1$	$\mathscr{T}_2$
(i)	Lower limit	Lower limit
(ii)	Lower limit	Upper limit
(iii)	Lower limit	Usual
(iv)	Lower limit	Cofinite
(v)	Lower limit	Cocountable
(vi)	Upper limit	Lower limit
(vii)	Upper limit	Upper limit
(viii)	Upper limit	Usual
(ix)	Upper limit	Cofinite
(x)	Upper limit	Cocountable
(xi)	Usual	Lower limit
(xii)	Usual	Upper limit
(xiii)	Usual	Usual
(xiv)	Usual	Cofinite
(xv)	Usual	Cocountable
(xvi)	Cofinite	Lower limit
(xvii)	Cofinite	Upper limit
(xviii)	Cofinite	Usual
(xix)	Cofinite	Cofinite
(xx)	Cofinite	Cocountable
(xxi)	Cocountable	Lower limit
(xxii)	Cocountable	Upper limit
(xxiii)	Cocountable	Usual
(xxiv)	Cocountable	Cofinite
(xxv)	Cocountable	Cocountable

TABLE 3. Continuity Between Topologies

17. Automatic Continuity of Certain Functions

When a certain hypothesis makes a function continuous, we describe that situation as the automatic continuity.

17.1. Proposition. *Let $f\colon (X,\mathscr{T}) \to (Y,\mathscr{T}')$ be a function between two topological spaces X and Y. We have the following.*

(1) If $\mathscr{T} = \mathscr{D}$, then f is continuous.
(2) If $\mathscr{T}' = \mathscr{I}$, then f is continuous.
(3) If f is constant, then f is continuous.

PROOF. ☞(1) Let $V \in \mathscr{T}'$. Then $f^{-1}(V) \subset X$. So, $f^{-1}(V) \in \mathscr{P}(X) = \mathscr{D} = \mathscr{T}$. Thus f is continuous.

☞(2) $V \in \mathscr{T}_2 = \mathscr{I} \Rightarrow V = \emptyset$ or $V = Y$. But then $f^{-1}(V) = \emptyset$ or $f^{-1}(V) = X$ both of which are in $\mathscr{T}$. Thus f is continuous.

☞(3) Let $f(x) = a$ for every $x \in X$. For $V \in \mathscr{T}'$,

$$f^{-1}(V) = \begin{cases} X, & \text{if } a \in V; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Again here both the sets are open in X. Thus f is continuous. □

17.2. Proposition. *An inclusion map is continuous.*

Proof. For a topological space $(X, \mathscr{T})$, $Y \subset X$ and $G \in \mathscr{T}$, we see that $\iota_Y^{-1}(G) = G \cap Y \in \mathscr{T}_Y$. Thus ι_Y is continuous. □

17.3. Corollary. *Restriction of a continuous map is continuous.*

Proof. Let X, Y be topological spaces, $f\colon X \to Y$ be a continuous function and $Z \subset X$. In one line, $f_{|_Z} = f \circ \iota_Z$ is a composition of two continuous functions, and hence, is continuous. However, we give a direct proof.

For an open set $V \subset Y$, ${f_{|_Z}}^{-1}(V) = f^{-1}(V) \cap Z$, which is open in Z as $f^{-1}(V)$ is open in X. This completes the proof. □

17.4. Proposition. *Let $\mathscr{T}, \mathscr{T}'$ be two topologies on a set X. Then the identity map $\iota_X\colon (X, \mathscr{T}) \to (X, \mathscr{T}')$ is continuous if and only if $\mathscr{T}' \subset \mathscr{T}$.*

Proof. Suppose $\iota_X\colon (X, \mathscr{T}) \to (X, \mathscr{T}')$ is continuous. Then $V \in \mathscr{T}' \Rightarrow V = \iota_X^{-1}(V) \in \mathscr{T}$. Thus $\mathscr{T}' \subset \mathscr{T}$.

Conversely, assume that $\mathscr{T}' \subset \mathscr{T}$. Let $V \in \mathscr{T}'$. Then $\iota_X^{-1}(V) = V \in \mathscr{T}' \subset \mathscr{T}$. Thus ι is continuous. This completes the proof. □

17.5. Proposition. *Let X be a topological space and $A \subset X$. Then the characteristic function χ_A of A is continuous if and only if A is a clopen subset of X.*

Proof. Suppose A is clopen set. Let $G \subset \mathbb{R}$ be an open set.

$0 \in G; 1 \in G \Rightarrow \chi_A^{-1}(G) = X;$

$0 \in G; 1 \notin G \Rightarrow \chi_A^{-1}(G) = X \setminus A;$

$0 \notin G; 1 \in G \Rightarrow \chi_A^{-1}(G) = A;$

$0 \notin G; 1 \notin G \Rightarrow \chi_A^{-1}(G) = \emptyset.$

Each of the above sets is open in X. Thus χ_A is continuous.

Conversely, suppose χ_A is continuous. Then $A = \chi_A^{-1}((0,2)) = \chi_A^{-1}(\{1\})$. Thus A is clopen. □

17.6. Definition. Let X, Y be topological spaces and $f\colon X \to Y$ be a function. We say that f is *locally constant* if for every $x \in X$ there is $U \in \mathscr{U}_x$ such that $f(U)$ is singleton.

17.7. Theorem. *Every locally constant function is continuous.*

Proof. Let X, Y be two topological spaces and $f\colon X \to Y$ be locally constant. Let $x \in X$ and $V \in \mathscr{U}_{f(x)}$. Let $U \in \mathscr{U}_x$ be such that $f(U) = f(x)$. But then $f(U) \subset V$. Thus f is continuous at x. Since x is arbitrary, it follows that f is continuous on X. □

Problems

126. Show that every real-valued continuous function on an indiscrete space is constant. Show that every function on a discrete space is locally constant.

127. Give an example of a locally constant function which is not constant.

128. Show that every real-valued locally constant function on $\mathbb{R}$ is constant.

18. Pasting Lemma

Many times, the function is defined in more than one piece. In that case, the continuity of each piece could be proved to assert the continuity of the whole function in certain situation.

18.1. Theorem. (**Pasting Lemma**) *Let X and Y be two topological spaces, $f\colon X \to Y$ be a function and $A, B \subset X$ be two open sets such that $A \cup B = X$. If $f_{|A}, f_{|B}$ are continuous, then f is continuous.*

Proof. Note that for a subset $U \subset Y$,

$$\begin{aligned} f^{-1}(U) &= f^{-1}(U) \cap X \\ &= f^{-1}(U) \cap (A \cup B) \\ &= (f^{-1}(U) \cap A) \cup (f^{-1}(U) \cap B) \\ &= (f^{-1}(U) \cap A) \cup (f^{-1}(U) \cap B) \\ &= ({f_{|A}}^{-1})(U) \cup ({f_{|B}}^{-1})(U). \end{aligned} \tag{18.1.1}$$

Now assume that U is open in Y. Then by the continuity of $f_{|A}$ and $f_{|B}$, $({f_{|A}}^{-1})(U)$ and $({f_{|B}}^{-1})(U)$ are open in A and B respectively. Since A and B are open in X, $({f_{|A}}^{-1})(U)$ and $({f_{|B}}^{-1})(U)$ are open in X by the Proposition 12.6. Thus $f^{-1}(U)$ is open in X. So f is continuous on X. □

Note that the continuity of f implies the continuity of $f_{|A}$ and $f_{|B}$ by the Corollary 17.3 irrespective of openness of A as well as B. The following is another version of pasting lemma.

18.2. Theorem. (**Pasting Lemma**) *Let X and Y be two topological spaces, $f\colon X \to Y$ be a function and $A, B \subset X$ be two closed sets such that $A \cup B = X$. If $f_{|A}, f_{|B}$ are continuous, then f is continuous.*

PROOF. Now let E be a closed subset of Y. Then computation as in (18.1.1) gives. $f^{-1}(E) = ({f_{|A}}^{-1})(E) \cup ({f_{|B}}^{-1})(E)$. Also by the continuity of $f_{|A}$ and $f_{|B}$, $({f_{|A}}^{-1})(E)$ and $({f_{|B}}^{-1})(E)$ are closed in A and B respectively. Since A and B are closed in X, $({f_{|A}}^{-1})(E)$ and $({f_{|B}}^{-1})(E)$ are closed in X. Thus $f^{-1}(E)$ is closed in X. Hence f is continuous on X. □

18.3. Example. If we define $f(x) = 0$ on $\mathbb{Q}$ and $f(x) = 1$ on $\mathbb{R} \setminus \mathbb{Q}$. Then f is not continuous on $\mathbb{R}$ even if the restrictions are.

The above statements of both version of the pasting lemma do not give the feeling of *pasting of two functions*. In fact, it can be paraphrased as following.

18.4. Theorem. (**Pasting Lemma**) *Let X and Y be two topological spaces, and A, B be two open (or both closed) subsets of X such that $A \cup B = X$. If $f\colon A \to Y, g\colon B \to Y$ are two continuous functions such that $f(x) = g(x)$ for all $x \in A \cap B$, then the function $h\colon X \to Y$ defined by*

$$h(x) = \begin{cases} f(x), & \text{if } x \in A; \\ g(x), & \text{if } x \in B \end{cases}$$

is continuous.

Problems

129. Show that the function $f\colon [0, \frac{\pi}{2}] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin x, & \text{if } x \in [0, \frac{\pi}{4}]; \\ \cos x, & \text{if } x \in [\frac{\pi}{4}, \frac{\pi}{2}] \end{cases}$$

is continuous. See the thick red portion of combination of two graphs in the Figure 14.

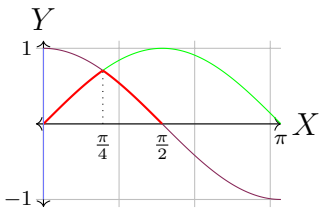


FIGURE 14. Pasting of Two Curves

130. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be two continuous functions such that $f(0) = g(0)$. Define

$$h(x) = \begin{cases} f(x), & \text{if } x \leq 0; \\ g(x), & \text{if } x \geq 0. \end{cases}$$

Show that h is continuous.

131. For natural number n, m, show that the function $f\colon [0,\infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^m, & \text{if } x \in [0,1]; \\ x^n, & \text{if } x \in [1,\infty) \end{cases}$$

is continuous.

19. Homeomorphism

19.1. Definition. Let X, Y be two topological spaces and $f\colon X \to Y$ be a function. We say that f is

(1) an *open map* if $f(G)$ is open whenever G is open in X.
(2) an *closed map* if $f(F)$ is closed whenever F is closed in X.
(3) a *homeomorphism* if f is one-one, onto, continuous and open map. In this case, we say that X *is homeomorphic to* Y, and write $X \cong Y$.

19.2. Example. A closed map need not be open. Indeed, a real-valued constant function on a topological space cannot be open.

19.3. Example. An open map need not be closed. Let $\mathbb{R}^2$ carry the usual topology as defined in the Example 8.17. Consider $f\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(x,y) = x$. Then f is an open map but it is not closed.

For $(a,b) \times (c,d) \subset \mathbb{R}^2$, $f((a,b) \times (c,d)) = (a,b)$. Now let $G \subset \mathbb{R}^2$ be open in $\mathbb{R}^2$. Then $G = \cup_i B_i$, where each $B_i = (a_i, b_i) \times (c_i, d_i)$. Then $f(G) = f(\cup_i B_i) = \cup_i f(B_i)$, which is open in $\mathbb{R}$.

On the other hand, let $A = \{(\frac{1}{n}, n) : n \in \mathbb{R}\}$. Then $f(A) = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed. But the set $G = (\mathbb{R} \times (\mathbb{R} \setminus \mathbb{N})) \cup (\cup_{n=1}^{\infty} [\mathbb{R} \setminus \{\frac{1}{n}\} \times (n - \frac{1}{2}, n + \frac{1}{2})])$ is open, and hence, $A = \mathbb{R}^2 \setminus G$ is closed. But $\pi_1(A) = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed.

19.4. Example ([1]). Let $S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, the unit circle, considered as a subspace of $\mathbb{R}^2$ and let

$$f\colon [0,1) \to S^1$$

be the map defined by $f(t) = (\cos 2\pi t, \sin 2\pi t)$. From the properties of trigonometric functions, it follows that f is bijective and continuous. However, the function $f^{-1}\colon S^1 \to [0,1)$ is not continuous since the image of open set $U = \left[0, \frac{1}{4}\right)$ under f is not open in S^1. This is because there is no open set V of $\mathbb{R}^2$ such that the point $p = f(0) \in V \cap S^1 \subset f(U)$.

[1] Courtesy Jay

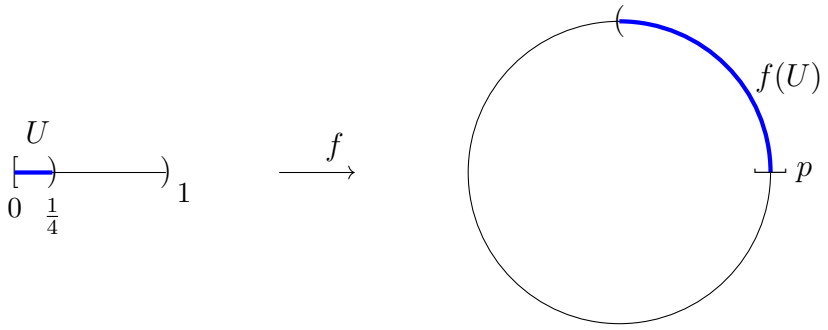


FIGURE 15. Bijective Continuous Non-homeomorphism

19.5. Theorem. *Let X, Y be two topological spaces and $f\colon X \to Y$ be a continuous open map. Let $x \in X$ and $y = f(x)$. Let $\mathscr{B}_x$ be a local base at x. Then $\mathscr{B}_y = \{f(B) : B \in \mathscr{B}_x\}$ is a local base at y.*

PROOF. For $V \in \mathscr{U}_y$, $f^{-1}(V) \in \mathscr{U}_x$. So, there is $B \in \mathscr{B}_x$ such that $B \subset f^{-1}(V)$. Hence $f(B) \in \mathscr{B}_y$ and $f(B) \subset V$. Thus $\mathscr{B}_y$ is a local base at y. □

19.6. Corollary. *A continuous open image of a first countable space is first countable.*

19.7. Example. An uncountable cofinite space is not first countable. Indeed, let X be an uncountable set with the cofinite topology $\mathscr{T}_f$. Suppose, if possible, that X is first countable. Let $x \in X$. Suppose $\mathscr{B}_x = \{B_n : n \in \mathbb{N}\}$ is a local base at x. But then $X \setminus B_n$ is finite for every n, and hence, $X \setminus \cap_{n=1}^{\infty} B_n = \cup_{n=1}^{\infty} X \setminus B_n$ is at most countable. Since X is uncountable, $\cap_{n=1}^{\infty} B_n$ must be uncountable. But, then for every $y \in \cap_{n=1}^{\infty} B_n \setminus \{x\}$, $x \in \{y\}'$, a contradiction to the fact that X is T_1. Thus X is not first countable.

19.8. Example. A continuous image of a first countable space need not be first countable. Indeed, let X be an uncountable set. Then $\iota\colon (X, \mathscr{D}) \to (X, \mathscr{T}_f)$ is a bijective continuous. Clearly, $(X, \mathscr{D})$ is first countable, but $(X, \mathscr{T}_f)$ is not first countable.

19.9. Corollary. *Homeomorphic image of a first countable space is first countable.*

19.10. Theorem. *Let X, Y be topological spaces, $f\colon X \to Y$ be a continuous onto open map. Let $\mathscr{B}$ be a base for X. Then $\{f(B) : B \in \mathscr{B}\}$ is a base for Y.*

PROOF. Clearly, $f(B)$ is open for every $B \in \mathscr{B}$ because f is an open map. Let V be an open subset of Y and $y \in V$. Then $f^{-1}(y) \subset f^{-1}(V)$ and $f^{-1}(V)$ is open in X. Choose $z \in f^{-1}(y)$. Then there is $B \in \mathscr{B}$ such that $z \in B \subset f^{-1}(V)$. So, $y \in f(B) \subset V$. Thus $\{f(B) : B \in \mathscr{B}\}$ is a base for Y. □

19.11. Corollary. *A continuous open image of a second countable space is second countable.*

19.12. Example. A continuous image of a second countable space need not be second countable. Indeed consider $\mathbb{R}$ with the usual topology. Then $\{(a,b) : a,b \in \mathbb{Q}, a<b\}$ is a countable base for $\mathbb{R}$, and hence $\mathbb{R}$ is second countable. However, $(\mathbb{R}, \mathscr{T}_f)$ is a continuous image of $\mathbb{R}$ with the usual topology. But $(\mathbb{R}, \mathscr{T}_f)$ is not second countable as it is not first countable.

We also prove directly, that an uncountable set X with the cofinite topology is not second countable. Suppose $\mathscr{B} = \{B_n : n \in \mathbb{N}\}$ is a base for $\mathscr{T}_f$. But then $X \setminus B_n$ is finite for every n, and hence, $\cup_{n=1}^{\infty} X \setminus B_n$ is at most countable. Thus $\cap B_n \neq \emptyset$, a contradiction. Thus X is not second countable.

19.13. Corollary. *Homeomorphic image of a second countable spaces is second countable.*

19.14. Proposition. *A composition of two homeomorphisms is a homeomorphism.*

Proof. Let X, Y, Z be topological spaces and $f\colon X \to Y$ and $g\colon Y \to Z$ be homeomorphisms. Clearly, $g \circ f$, being composition of two bijections, is a bijection. Also continuity of f, g ensures the continuity of $g \circ f$ and the continuity of f^{-1}, g^{-1} ensures the continuity of $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. □

19.15. Proposition. *Let X, Y, Z be topological spaces.*
(1) $X \cong X$
(2) $X \cong Y \Rightarrow Y \cong X$.
(3) $X \cong Y, Y \cong Z \Rightarrow X \cong Z$.

Proof. ☞(1) Indeed $\iota\colon X \to X$ defined by $\iota(x) = x$, $(x \in X)$, is a homeomorphism.

☞(2) Let $f\colon X \to Y$ be a homeomorphism. Hence f is bijective. Thus $f^{-1}\colon Y \to X$ is defined and bijective. Since f is continuous, f^{-1} is open. Since f is open, f^{-1} is continuous. Thus f^{-1} is a homeomorphism.

☞(3) Follows from the Proposition 19.14. □

19.16. Proposition. *If $f\colon X \to Y$ is a homeomorphism and $Z \subset X$, then $f_{|Z}\colon Z \to f(Z)$ is a homeomorphism.*

Proof. Clearly, $f_{|Z}\colon Z \to f(Z)$ is bijective. Since f and f^{-1} are continuous, their restrictions $f_{|Z}$ and $(f_{|Z})^{-1} = f^{-1}{}_{|f(Z)}$ are continuous. Hence the result follows. □

It is natural to ask the following question.

Question: Let X, Y be two topological spaces such that $f\colon X \to Y$ is a continuous bijection. Is f^{-1} continuous? Further suppose $g\colon Y \to X$ is a continuous bijection. Is $X \cong Y$? The following answers this negatively.

19.17. Example. Consider $X = (0, 1]$, $Y = (0, 1)$ with the usual topology. Using the Proposition 12.10, we define the discrete extension topology of X and Y to $\mathbb{R}$. That is

$$\mathscr{T}^{X\mathscr{D}} = \{G \subset \mathbb{R} : G \cap X \text{ is open in } X\}$$
$$\mathscr{T}^{Y\mathscr{D}} = \{G \subset \mathbb{R} : G \cap Y \text{ is open in } Y\}.$$

Also let $\mathbb{R}_X = (\mathbb{R}, \mathscr{T}^{X\mathscr{D}})$ and $\mathbb{R}_Y = (\mathbb{R}, \mathscr{T}^{Y\mathscr{D}})$. We show that the formula $f(x) = \frac{x}{2}$, $(x \in \mathbb{R})$, defines a continuous bijection from $\mathbb{R}_X$ to $\mathbb{R}_Y$ and from $\mathbb{R}_Y$ to $\mathbb{R}_X$. But $f^{-1}(x) = 2x$ is not continuous. Note that the following sets are basic open subsets of $\mathbb{R}_Y$ and except $\{1\}$ all other in the list are basic open subsets of $\mathbb{R}_X$ too.

(1) (a, b), $0 \le a < b \le 1$;
(2) $\{x\}$, $x \notin (0, 1)$.

We first show that $f^{-1}(B)$, where B is of the above type, is open in $\mathbb{R}_X$ as well as $\mathbb{R}_Y$. Indeed since $f^{-1}((a, b)) \cap X = (2a, 2b) \cap X$ is open in X and $f^{-1}((a, b)) \cap Y = (2a, 2b) \cap Y$ is open in Y, we conclude that $f^{-1}(a, b)$ is open in X as well as Y. Also, for $x \notin (0, 1)$, $f^{-1}(\{x\}) \cap X = \{2x\} \cap X = \emptyset$ is an open subset of X and $f^{-1}(\{x\}) \cap Y = \{2x\} \cap Y = \emptyset$ is an open subset of Y. Since all basic open subsets of $\mathbb{R}_Y$ are exhausted, we conclude that $f\colon \mathbb{R}_X \to \mathbb{R}_Y$ is continuous.

Additionally, $\mathbb{R}_X$ has the following type of basic open sets.

(3) $(b, 1]$, $0 \le b < 1$.

Albeit $f^{-1}((b, 1]) \cap Y = (2b, 2] \cap Y = (2b, 1)$, an open set in Y. Thus $f\colon \mathbb{R}_Y \to \mathbb{R}_X$ is continuous.

On the other hand, $\{1.2\}$ is open in $\mathbb{R}_X$ as well as $\mathbb{R}_Y$, but $f(\{1.2\}) = \{0.6\}$ is neither open in $\mathbb{R}_X$ nor open in $\mathbb{R}_Y$. Thus f^{-1} is not continuous, and so, f is not a homeomorphism.

19.18. Definition. Let X, Y be two topological spaces and $f\colon X \to Y$ be a function. If $f\colon X \to f(X)$, is a homeomorphism, then we say that f is an *embedding*. We denote this by $X \overset{f}{\hookrightarrow} Y$.

19.19. Example. For $m \in \mathbb{R} \setminus \{0\}$ and $c \in \mathbb{R}$, the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = mx + c$, $(x \in \mathbb{R})$, is a homeomorphism.

Of course the continuity of f is at once because

$$f^{-1}((a, b)) = \begin{cases} (\frac{a-c}{m}, \frac{b-c}{m}), & \text{if } m > 0; \\ (\frac{b-c}{m}, \frac{a-c}{m}), & \text{if } m < 0, \end{cases}$$

each of which is an open set. Also f is bijective and the form $f^{-1}(y) = \frac{y}{m} - \frac{c}{m}$ guarantees the continuity of f^{-1}.

The above, in conjunction with the Proposition 19.16, gives the following.

19.20. Example. The functions f defined by $f(x) = (b-a)x + a$, as shown in the Figure 16(a) and h defined by $h(x) = (b-a)x^2 + a$, as shown in the Figure 16(b), define homeomorphisms from (1) $[0,1] \to [a,b]$ (2) $(0,1) \to (a,b)$, (3) $[0,1) \to [a,b)$ and (4) $(0,1] \to (a,b]$. Also, the function g defined by $g(x) = (a-b)x + b$, as shown in the Figure 16(c), defines homeomorphisms from (1) $[0,1] \to [a,b]$ (2) $(0,1) \to (a,b)$, (3) $[0,1) \to (a,b]$ and (4) $(0,1] \to [a,b)$.

(a) $(b-a)x + a$ (b) $(b-a)x^2 + a$ (c) $(a-b)x + b$

FIGURE 16. Homeomorphisms Between Intervals

Based on the school mathematics, we accept the continuity of the tan and $\tan^{-1}$ functions on appropriate domain and codomain without proving it. Consequently, we have the following homeomorphisms.

19.21. Example. $f\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ defined by $f(t) = \tan t$, $(t \in -\frac{\pi}{2}, \frac{\pi}{2})$, is a homeomorphism.

19.22. Example. $f\colon (-1,1) \to \mathbb{R}$, defined by $f(x) = \tan(\frac{\pi x}{2})$, $(x \in (-1,1))$, is a homeomorphism.

19.23. Example. $f\colon \mathbb{R}_\ell \to \mathbb{R}_u$ defined by $f(x) = -x$, $(x \in \mathbb{R})$, is a homeomorphism.

The following is a straightforward verification.

19.24. Theorem. *Let X, Y be two topological spaces and $f\colon X \to Y$ be a homeomorphism. Let $A \subset X$ and $x \in X$. Then the following hold.*

(1) $f^{-1}\colon Y \to X$ is a homeomorphism.
(2) A is open in X if and only if $f(A)$ is open in Y.
(3) A is closed in X if and only if $f(A)$ is closed in Y.
(4) $\mathscr{B}$ is a base for X if and only if $f(\mathscr{B}) = \{f(B) : B \in \mathscr{B}\}$ is a base for Y.
(5) $\mathscr{B}_x$ is a local base at x if and only if $f(\mathscr{B}_x) = \{f(B) : B \in \mathscr{B}_x\}$ is a local base at $f(x)$.
(6) $x \in A^\circ$ if and only if $f(x) \in f(A)^\circ$.
(7) $x \in A'$ if and only if $f(x) \in f(A)'$.
(8) $x \in \overline{A}$ if and only if $f(x) \in \overline{f(A)}$.
(9) $x \in \mathrm{bd}(A)$ if and only if $f(x) \in \mathrm{bd}(f(A))$.

Problems

132. Let X and Y be two homeomorphic spaces and $x \in X$. Let $f\colon X \to Y$ be homeomorphism. Show that $X \setminus \{x\}$ is homeomorphic to $Y \setminus \{f(x)\}$.

133. Prove that the two topological spaces X and Y in the Example 19.17 are not homeomorphic to each other.

134. Verify which of the following functions are homeomorphisms from their domains to codomains.

(1) $f\colon (0,1) \to (1,\infty)$, $f(x) = \frac{1}{x}$, $(x \in (0,1))$.
(2) $f\colon (0,\frac{\pi}{2}) \to (0,1)$, $f(x) = \sin x$, $(x \in (0,\frac{\pi}{2}))$.
(3) $f\colon [0,\frac{\pi}{2}] \to [0,1]$, $f(x) = \sin x$, $(x \in [0,\frac{\pi}{2}])$.
(4) $f\colon [-\pi,\pi] \to [-1,1]$, $f(x) = \sin x$, $(x \in [-\pi,\pi])$.
(5) $f\colon (0,1) \to (5,7)$, $f(x) = 2x^2 + 5$, $(x \in (0,1))$.
(6) $f\colon (0,1) \to (5,7)$, $f(x) = 2x^7 + 5$, $(x \in (0,1))$.

135. Define $f\colon [0,1] \to [0,1]$ by

$$f(x) = \begin{cases} \frac{x}{2}, & \text{if } 0 \le x \le \frac{3}{4}; \\ mx + c, & \text{if } \frac{3}{4} \le x \le 1. \end{cases}$$

Find the values of m, c so that f becomes a homeomorphism. Are the values of m, c unique? Justify.

136. Let X, Y be topological spaces, $A \subset X$ and $h\colon X \to Y$ be a homeomorphism. Show that

(1) $h(A^\circ) = h(A)^\circ$.
(2) $\overline{h(A)} = h(\overline{A})$.
(3) $h(A') = h(A)'$.
(4) $h(\mathrm{bd}(A)) = \mathrm{bd}(h(A))$.

137. For $X = [0,1]$ and $n \in \mathbb{N}$, define $f_n\colon X \to f(X)$, $f(x) = x^n$, $(x \in X)$.

(1) Is each f_n a homeomorphism?
(2) Does your conclusion hold if $X = [-1,1]$? if $X = [1,2]$? if $X = [-1,0]$? Justify.

138. Prove the Theorem 19.24. Also verify if any of the properties (1)–(9) implies that f is a homeomorphism.

20. Limit

The concept of continuity is introduced through limit in the school. However later on the limit is dropped in Point Set Topology and the continuity is introduced in the language of open sets and their inverse images. In this section, we define the concept of limit for functions between topological spaces and recapture the continuity using limits.

20.1. Definition. Let X, Y be two topological spaces, $f\colon X \to Y$ be a function, $a \in X$ and $\ell \in Y$. We say that $f(x) \to \ell$ as $x \to a$ (read as $f(x)$ tends to ℓ as x tends to a) if the following hold.

(1) If x is an isolated point of X, then $f(x) = \ell$.
(2) If x is not an isolated point of X, then for every $V \in \mathscr{U}_\ell$ there is $U \in \mathscr{U}_x$ such that $f(U \setminus \{x\}) \subset V$.

In this case, we also write $\lim\limits_{x\to a} f(x) = \ell$.

It is clear from the definition of the limit of a function that our definition coincides with the usual definition of limit in the classical case.

20.2. Example. Let X, Y be a topological spaces, $a \in X$ and $f\colon X \to Y$ be a constant function, say $f(x) = c$, $(x \in X)$. Let $\ell \in Y$. If every neighbourhood V of ℓ contains c, then $\lim\limits_{x\to a} f(x) = \ell$. In particular, $\lim\limits_{x\to a} f(x) = c$. Thus limit of a constant function exists at every point, irrespective of topologies on domain and codomain of the function. However, limit need not be unique.

Important Note

For a function $f\colon X \to Y$ and $a \in X$, there could be $l \neq m$ in Y such that $\lim\limits_{x\to a} f(x) = \ell$ and $\lim\limits_{x\to a} f(x) = m$. This should not be used to infer incorrectly, that $\ell = m$.

20.3. Example. Let $X = \mathbb{R}$ with the indiscrete topology, $a \in X$ and $Y = \{0, 1\}$ with $\mathscr{T}_0 = \{\emptyset, Y, \{0\}\}$, the Sierpińsky topology. Let $f = \chi_{\mathbb{Q}}$, the characteristic function of $\mathbb{Q}$. Then $\lim\limits_{x\to a} f(x) = 1$. But 0 is not the limit of f at any point. On the other hand, let $g\colon X \to Y$ be defined as $g(x) = 0$, $(x \in X)$. Then $\lim\limits_{x\to a} g(x) = 0$ as well as $\lim\limits_{x\to a} g(x) = 1$.

20.4. Example. Let X be a discrete space, Y be a topological space and $f\colon X \to Y$ be a function. Then for every $a \in X$, $\lim\limits_{x\to a} f(x) = f(a)$. Indeed, this is because every point of X is an isolated point.

20.5. Example. Let X be an indiscrete space, Y be a topological space, $f\colon X \to Y$ be a function, $a \in X$ and $\ell \in Y$. Then $\lim\limits_{x\to a} f(x) = \ell$ if and only if every neighbourhood of ℓ contains $f(X \setminus \{a\})$. In particular, limit of a constant function exists at each point.

20.6. Example. Let X be a topological space, Y be an indiscrete space, $f\colon X \to Y$ be a function, $a \in X$ and $\ell \in Y$. If a is not an isolated point of X, then $\lim\limits_{x\to a} f(x) = \ell$ because Y is the only neighbourhood of ℓ and $f(X) \subset Y$. However if a is an isolated point of X, then $\lim\limits_{x\to a} f(x) = \ell$ if and only if $f(a) = \ell$.

20.7. Theorem. *Let X, Y be a topological spaces, $f\colon X \to Y$ be a function and $a \in X$. If $\lim\limits_{x\to a} f(x) = \ell$ as well as $\lim\limits_{x\to a} f(x) = m$, then every neighbourhood of ℓ intersects every neighbourhood of m.*

PROOF. If a is an isolated point, then $\lim_{x\to a} f(x) = f(a)$ and hence $\ell = f(a) = m$. Thus every neighbourhood of ℓ is a neighbourhood of $f(a)$ which in turn is a neighbourhood of m. Next we assume that a is not an isolated point of X. Suppose U, V are neighbourhoods of ℓ and m respectively. By the hypothesis, there are two neighbourhoods U_1, V_1 of a such that $f(U_1 \setminus \{a\}) \subset U$ and $f(V_1 \setminus \{a\}) \subset V$. If $U_1 \cap V_1 = \{a\}$, then a is an isolated point of X which is not the case. Thus there should be $b \in U_1 \cap V_1$. But then $f(b) \in U \cap V$. This completes the proof. □

20.8. Theorem. *Let X, Y be topological spaces and $f\colon X \to Y$ be a function. Then f is continuous at a if and only if $\lim_{x\to a} f(x) = f(a)$.*

PROOF. If a is an isolated point of X, then the result follows easily. So, we assume that a is not an isolated point of X. Suppose f is continuous at a let V be a neighbourhood of $f(a)$, then there is a neighbourhood of U of a such that $f(U) \subset V$, and hence $f(U \setminus \{a\}) \subset V$.

Conversely, suppose $\lim_{x\to a} f(x) = f(a)$. Let V be a neighbourhood of $f(a)$. Then there is a neighbourhood U of a such that $f(U \setminus \{a\}) \subset V$. Since, $f(a) \in V$, it follows that $f(U) \subset V$. Consequently, f is continuous at a. □

20.9. Example. Let X be an infinite set, $a, b \in X$ and $\iota\colon X \to X$ be the identity map.

(1) $\lim_{x\to a} \iota(x) = b$ with respect to the cofinite topology on X.
(2) $\lim_{x\to a} \iota(x) = b$ with respect to the cocountable topology on X.
(3) $\lim_{x\to a} \iota(x) = b$ with respect to the indiscrete topology on X.
(4) $\lim_{x\to a} \iota(x) = a$ with respect to the discrete topology on X.

In (1) and (2) let V be a neighbourhood of b and $U = V \cup \{a\}$. Then U is a neighbourhood of a. Also, $\iota(U \setminus \{a\}) \subset \iota(V) = V$. This proves the corresponding claims.

☞(3) is obvious as X is the only neighbourhood of a as well as b.

☞(4) is obvious because each point is an isolated point of X.

CHAPTER 4

Categorial Methods

21. Weak and Quotient Topologies

21.1. Lemma. *Let X be a set and $(Y, \mathscr{T})$ be a topological space. Define*

$$\mathscr{T}_w(f) = \{f^{-1}(V) : V \in \mathscr{T}\}.$$

Then

(1) $\mathscr{T}_w(f)$ is a topology on X.
(2) $f\colon (X, \mathscr{T}_w(f)) \to (Y, \mathscr{T})$ is continuous.
(3) If $\mathscr{T}'$ is a topology on X such that $f\colon (X, \mathscr{T}') \to (Y, \mathscr{T})$ is continuous, then $\mathscr{T}_w(f) \leq \mathscr{T}'$.

All these three together can be described as

(4) $\mathscr{T}_w(f)$ is the weakest topology on X in which f is continuous.

Proof. $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$. Let $\{f^{-1}(V_i) : i \in I\} \subset \mathscr{T}_w(f)$ with each V_i open in Y. So, $\cup_{i\in I} f^{-1}(V_i) = f^{-1}(\cup_{i\in I} V_i) \in \mathscr{T}_w(f)$. Finally, for $f^{-1}(V_1), f^{-1}(V_2) \in \mathscr{T}_w(f)$ with V_1, V_2 open in Y, $f^{-1}(V_1) \cap f^{-1}(V_2) = f^{-1}(V_1 \cap V_2) \in \mathscr{T}_w(f)$. This proves (1).

☞(2) follows from the the definition of $\mathscr{T}_w(f)$.

Let $\mathscr{T}'$ be a topology on X inSo, which f is continuous and $U \in \mathscr{T}_w(f)$. Hence $U = f^{-1}(V)$ for some open set V in Y. By the continuity of $f\colon (X, \mathscr{T}') \to Y$, $U = f^{-1}(V) \in \mathscr{T}'$. So $\mathscr{T}_w(f) \leq \mathscr{T}'$, making $\mathscr{T}_s(f)$ the weakest topology on X in which f is continuous. This proves (3). □

Now we consider a family of functions defined on X.

21.2. Theorem. *Let X be a set and $\Gamma = \{(X_i, \mathscr{T}_i, f_i) : f_i \colon X \to X_i, i \in I\}$ be a family of topological spaces with associated functions defined on X. Define*

$$\mathscr{S} = \mathscr{S}_w(\Gamma) = \{f_i^{-1}(V_i) : V_i \in \mathscr{T}_i, i \in I\}.$$

Let $\mathscr{T} = \mathscr{T}_w(\Gamma)$ be a topology on X generated by the subbase $\mathscr{S}$. Then $\mathscr{T}$ is the weakest topology on X in which each f_i is continuous.

Proof. Clearly, $\mathscr{S} \subset \mathscr{T}$, and hence, for every $i \in I$ and $V_i \in \mathscr{T}_i$, $f_i^{-1}(V_i) \in \mathscr{S} \subset \mathscr{T}$. Thus f_i is continuous for every $i \in I$. Finally, let $\mathscr{T}'$ be a topology on X such that each $f_i \colon (X, \mathscr{T}') \to (X_i, \mathscr{T}_i)$ is continuous. Let $S \in \mathscr{S}$. Then $S = f_i^{-1}(V_i)$ for some $i \in I$ and $V_i \in \mathscr{T}_i$. By the continuity, of $f_i \colon (X, \mathscr{T}') \to (X_i, \mathscr{T}_i)$, $S \in \mathscr{T}_i$. That is, $\mathscr{S} \subset \mathscr{T}'$. But by the Proposition 9.3, $\mathscr{T}$ is the smallest topology containing $\mathscr{S}$. Hence $\mathscr{T} \subset \mathscr{T}'$. □

21.3. Definition. Let X be a set and $\Gamma = \{(X_i, \mathscr{T}_i, f_i) : f_i \colon X \to X_i, i \in I\}$ be a family of topological spaces with associated functions defined on X. Then the topology $\mathscr{T}_w(\Gamma)$ in the Theorem 21.2 is called the *weak topology generated by* Γ.

21.4. Theorem. *Let a set X carry the weak topology $\mathscr{T}_w(\Gamma)$ given by a family $\Gamma = \{(X_i, \mathscr{T}_i, f_i) : f_i \colon X \to X_i, i \in I\}$ of topological spaces with associated functions defined on X. Let $(Y, \mathscr{T})$ be a topological space and $f \colon Y \to X$ be a function. Then f is continuous if and only if $f_i \circ f$ is continuous for every $i \in I$.*

Proof. Since composition of continuous functions is continuous, it follows that $f_i \circ f$ is continuous for every $i \in I$, whenever so is f.

Conversely, suppose $f_i \circ f$ is continuous for every $i \in I$. Let $S \in \mathscr{S}_w(\Gamma)$, defined in the Theorem 21.2. Then $S = f_i^{-1}(V)$ for some $V \in \mathscr{T}_i$. But by the continuity of $f_i \circ f$, $f^{-1}(S) = f^{-1}(f_i^{-1}(V)) = (f_i \circ f)^{-1}(V)$ is open in Y. Thus f is continuous by the Theorem 16.7. □

21.5. Proposition. *Let X be a set and Y be a topological space. Let $f \colon X \to Y$ be a bijective function. Then $f \colon (X, \mathscr{T}_w(f)) \to (Y, \mathscr{T})$ is a homeomorphism.*

Proof. The only point required to be proved is that f is open. Let $U \in \mathscr{T}_w(f)$. So, there is $V \in \mathscr{T}$ such that $U = f^{-1}(V)$. Since f is a bijection, $V = f(U)$. Thus f is open, and hence, f is a homeomorphism. □

21.6. Example. Let X be a set, Y be a topological space and $f \colon X \to Y$ be a constant map. Then $\mathscr{T}_w(f)$ is the indiscrete topology.

Indeed, let $f(X) = \{a\}$. For any $G \in \mathscr{T}$, $f^{-1}(G) = \emptyset$ if $a \notin G$ and $f^{-1}(G) = X$ if $a \in G$. Thus $\mathscr{T}_w(f) = \{\emptyset, X\} = \mathscr{I}$.

21.7. Example. Let X be a set, Y be an indiscrete space and $f\colon X \to Y$ be a function. Then $\mathscr{T}_w(f)$ is the indiscrete topology.

Indeed, let $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$ and these are the only open sets.

21.8. Example. Let X be a set and Y be a discrete space. Let $f\colon X \to Y$ be a function. Then the weak topology on X given by f is discrete if and only if f is one-one.

Indeed suppose f is one-one. Let $x \in X$, then $\{f(x)\}$ is open in Y, and hence, $\{x\} = f^{-1}(f(\{x\}))$ is open in X. On the other hand suppose f is not one-one. Let x_1, x_2 be two distinct points of X such that $f(x_1) = f(x_2) = y$. Let $G \subset X$ be an open set containing x_1. So, there is an open set V in Y such that $G = f^{-1}(V)$. But then $x_1 \in G \Rightarrow y \in V \Rightarrow x_2 \in G$. Thus every open set containing x_1 also contains x_2. Thus $\{x_1\}$ is not open in X.

21.9. Example. Let X be a set and $\Gamma = \{\delta_x : x \in X\}$ be the family of all point evaluations on X. Then the weak topology defined by Γ is the discrete topology. Indeed, consider $x \in X$. Then $\{x\} = \delta_x^{-1}(\{1\})$ is open in X, Thus each singleton is open in X.

21.10. Example. Consider the weak topology on $\mathbb{R}$ defined by the family $\Gamma = \{\chi_{[a,b)} : -\infty < a < b \leq \infty\}$ of characteristic functions. Then Γ determines the lower limit topology on $\mathbb{R}$. Indeed the claim follows by noting that $\mathscr{B} = \{[a, b) : -\infty < a < b \leq \infty\}$ is a base for the lower limit topology on $\mathbb{R}$.

Problems

139. Let $f\colon [0, 5] \to \mathbb{R}$ be defined by $f(x) = x^2$, $(x \in [0, 5])$. Show that the weak topology on $[0, 5]$ given by f is the same as the usual topology on $[0, 5]$.

140. Let $f\colon [-5, 5] \to \mathbb{R}$ be defined by $f(x) = x^2$, $(x \in [-5, 5])$. Show that the weak topology on $[-5, 5]$ given by f is not the same as the usual topology on $[-5, 5]$.

141. Let $f\colon \mathbb{R} \to \mathbb{R}_\ell$ be defined by $f(x) = -x$, $(x \in \mathbb{R})$. Show that the weak topology on $\mathbb{R}$ given by f is the same as the upper limit topology on $\mathbb{R}$.

142. Show that the weak topology on $\mathbb{R}$ given by the characteristic functions $\{\chi_{[a,b)} : -\infty < a < b < \infty\}$ coincides with the lower limit topology.

143. Let $f\colon [0, 2\pi] \to \{z : |z| = 1\}$ be defined by $f(x) = e^{ix}$, $(x \in [0, 2\pi])$. Show that the weak topology on $[0, 2\pi]$ given by f is not the same as the usual topology on $[0, 2\pi]$.

144. Let $f\colon (0, 2\pi] \to \{z : |z| = 1\}$ be defined by $f(x) = e^{ix}$, $(x \in (0, 2\pi])$. Show that the weak topology on $(0, 2\pi]$ given by f is not the same as the usual topology on $(0, 2\pi]$.

145. Let X be a topological space and $Y \subset X$. Show that the subspace topology on Y is the weak topology given by the $\iota_Y\colon Y \to X$.

146. Let $\{X_i : i \in I\}$ and $\{\Lambda_i : i \in I\}$ be families of sets and X be a set. For each $i \in I$, let $\{X_{i_\alpha} : \alpha \in \Lambda_i\}$ be a family of topological spaces. For each $\alpha \in \Lambda_i$, let $f_{i_\alpha}\colon X_i \to X_{i_\alpha}$ be a function. Also for each $i \in I$, let $f_i\colon X \to X_i$ be a function. Let X_i carry the weak topology given by $\{f_{i_\alpha} : \alpha \in \Lambda_i\}$. Show that the weak topology on X given by $\{f_i : i \in I\}$ is the same as the weak topology given by $\{f_{i_\alpha} \circ f_i : i \in I, \alpha \in \Lambda_i\}$.

147. Let X be a set and $\{A_1, A_2, \ldots, A_n\}$ be a partition of X. Define $f\colon X \to \mathbb{R}$ by $f(x) = \sum_{k=1}^{n} k\chi_{A_k}$, $(x \in X)$. Determine the weak topology on X given by f.

148. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x$, $(x \in \mathbb{R})$. Mention a nonempty proper open subset of $\mathbb{R}$ in the weak topology given by f.

149. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$, $(x \in \mathbb{R})$. Show that the weak topology on $\mathbb{R}$ is the same as the usual topology. What can you say if we replace x^3 by x^2?

150. For $j = 1, 2$, define $\pi_j\colon \mathbb{R}^2 \to \mathbb{R}$ by $\pi_j(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_j$, $((\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^2)$. Show that the weak topology on $\mathbb{R}^2$ is the same as the usual topology on $\mathbb{R}^2$.

21.11. Proposition. *For a topological space $(X, \mathscr{T})$, a set Y and a function $f\colon X \to Y$, define*

$$\mathscr{T}' = \{G \subset Y : f^{-1}(G) \in \mathscr{T}\}. \tag{21.11.1}$$

Then $\mathscr{T}'$ is the strongest topology on Y such that f is continuous. Also, $Y \setminus f(X)$ carries the discrete topology as a subspace.

Proof. Clearly, $f^{-1}(\emptyset) = \emptyset$ and $f^{-1}(Y) = X$ are open in X. So, $\emptyset, Y \in \mathscr{T}'$.

Let $\{G_i : i \in I\} \subset \mathscr{T}'$. Then $f^{-1}(G_i) \in \mathscr{T}$ for every i. Hence $\cup_i f^{-1}(G_i) \in \mathscr{T}$. Hence $f^{-1}(\cup_i G_i) = \cup_i f^{-1}(G_i) \in \mathscr{T}$. Thus $\cup_i (G_i) \in \mathscr{T}'$.

Let $G, H \in \mathscr{T}'$. Then $f^{-1}(G), f^{-1}(H) \in \mathscr{T}$. So, $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H) \in \mathscr{T}$, which, in turn, confirms that $G \cap H \in \mathscr{T}'$.

The definition of $\mathscr{T}'$ explicitly boasts of continuity of f.

Now let $\mathscr{T}_1$ be a topology on Y such that $f\colon (X, \mathscr{T}) \to (Y, \mathscr{T}_1)$ is continuous. Let $G \in \mathscr{T}_1$. By the assumed continuity of f, $f^{-1}(G) \in \mathscr{T}$. Hence $G \in \mathscr{T}'$.

Finally, for any $G \subset Y \setminus f(X)$, $f^{-1}(G) = \emptyset$, and hence, G is open in Y as well as in $Y \setminus f(X)$. This completes the proof. □

21.12. Proposition. *Let X, Y be two topological spaces and $f\colon X \to Y$ be a function. Let $f(X) \subset Z \subset Y$. Then $f\colon X \to Y$ is continuous if and only if $f\colon X \to Z$ is continuous.*

Proof. Let $f\colon X \to Y$ be continuous and $W \subset Z$ be open. Then there is an open set V in Y such that $W = V \cap Z$. Since $f^{-1}(W) = f^{-1}(V)$, the continuity of $f\colon X \to Z$ is at once.

Conversely, suppose $f\colon X \to Z$ is continuous and $V \subset Y$ be open in Y. But then $V \cap Z$ is open in Z. Again since $f^{-1}(V) = f^{-1}(V \cap Z)$, the continuity of $f\colon X \to Y$ follows. This completes the proof. □

Since the topology on $Y \setminus f(X)$ is discrete, it is customary to assume while defining the quotient topology, that f is onto.

21.13. Definition. Let $(X, \mathscr{T})$ be a topological space, Y be a set and $f\colon X \to Y$ be an onto function. The topology defined by (21.11.1) is called the *quotient topology* defined by f and it is denoted by $\mathscr{T}_q(f)$. Also, Y is called a *quotient space* of X and is denoted by X/f. The function f is called the *quotient map*.

The case of many functions replacing one function in the set up of quotient space is very easy.

21.14. Definition. Let X be a set and $\Gamma = \{(X_i, \mathscr{T}_i, f_i) : i \in I\}$ be a family of topological spaces $(X_i, \mathscr{T}_i)$, each with one onto function $f_i \colon X_i \to X$. Then the strongest topology $\mathscr{T}_q(\Gamma)$ on X with respect to which each f_i is continuous is called the *quotient of X* generated by Γ.

21.15. Proposition. *Let X be a set and $\Gamma = \{(X_i, \mathscr{T}_i, f_i) : i \in I\}$ be a family of topological spaces $(X_i, \mathscr{T}_i)$, each with one function $f_i \colon X_i \to X$. Then*

$$\mathscr{T}_q(\Gamma) = \{G \subset X : f_i^{-1}(G) \in \mathscr{T}_i \text{ for all } i \in I\} = \cap_i \mathscr{T}_q(f_i).$$

We leave the proof for the reader.

The following characterizes the closed subsets of a quotient.

21.16. Proposition. *Let X be a set and $\Gamma = \{(X_i, \mathscr{T}_i, f_i) : i \in I\}$ be a family of topological spaces $(X_i, \mathscr{T}_i)$, each with one function $f_i \colon X_i \to X$. and $\mathscr{T}_q$ be the quotient topology on X. Then $F \subset X$ is closed in X if and onl if $f_i^{-1}(F)$ is closed in X_i for each $i \in I$.*

Proof. Note that for each $i \in I$, $X_i \setminus f_i^{-1}(F) = f_i^{-1}(X \setminus F)$. So, F is closed in X if and only if $X \setminus F$ is open in X if and only if $X_i \setminus f_i^{-1}(F)$ is open in each X_i if and only if $f_i^{-1}(F)$ is closed in X_i for each $i \in I$. □

21.17. Theorem.

(1) A continuous surjective open map is a quotient map.
(2) A continuous surjective closed map is a quotient map.

Proof. Let $(X, \mathscr{T}), (Y, \mathscr{T}')$ be two topological spaces and $f \colon X \to Y$ be a continuous surjective map. Clearly, by the Proposition 21.11, $\mathscr{T}' \leq \mathscr{T}_q(f)$. So, in each case, we have to show that $\mathscr{T}_q(f) \leq \mathscr{T}'$.

☞(1) Assume that f is an open map. Now $G \in \mathscr{T}_q(f) \Rightarrow f^{-1}(G) \in \mathscr{T} \Rightarrow G = f(f^{-1}(G)) \in \mathscr{T}'$. So, $\mathscr{T}_q(f) \leq \mathscr{T}'$

☞(2) Assume that f is an closed map. Let $G \in \mathscr{T}'$ and $F = Y \setminus G$. By the continuity of f, $f^{-1}(F)$ is $\mathscr{T}$-closed. Hence $f(f^{-1}(F))$ is $\mathscr{T}'$-closed, and hence, $G \in \mathscr{T}'$. So, $\mathscr{T}_q(f) \leq \mathscr{T}'$. □

21.18. Example. A quotient map need not be closed or open, in general.

Indeed, consider $\mathbb{R}$ with the usual topology and $X = \{a, b\}$ with $f \colon \mathbb{R} \to X$ defined by $f(x) = \begin{cases} a, & \text{if } x \in [0,1]; \\ b, & \text{if } x \notin [0,1]. \end{cases}$ Then $\mathscr{T}_q(f) = \{\emptyset, X, \{b\}\}$ and $f([7,8]) = \{b\}$

is not closed as $\{a\}$ is not open. Also, $f((0,1)) = \{a\}$, which is not open. Thus f is neither closed, nor open.

Now we give an equivalent frequently used construction of quotient. Let $f\colon X \to Y$ be a surjective function. Then by the Theorem 5.11, $\bar{f}\colon X/f \to Y$ is a one to one correspondence. Thus we identify Y with X/f and instead of giving topology on Y, we give the quotient topology on X/f. Since the only effect of Y and f here is to yield an equivalence relation, we do not have any rôle of Y and f in the final version. Thus all in all, we use an equivalence relation. Not only this, but most of the time, in an equivalence relation, the partition is mentioned only using which points of X are identified. Having done that, Y is obtained as a byproduct. The following example makes it clear.

21.19. Example. Identifying the end points of the interval $[0,1]$, we obtain a circle as a quotient.

Here the partition of $[0,1]$ consists of all singletons except the points $0, 1$, which are identified with each other in the sense that, for $x, y \in [0,1]$, we define $x \sim y$ if either $x = y$ or if $x, y \in \{0,1\}$. Then the quotient $X/\sim \cong S^1 = \{(x,y) : x^2 + y^2 = 1\}$. That is, the quotient space $X/\sim$ is homeomorphic to the sphere S^1.

FIGURE 17. Circle as Quotient

Indeed, define $t \in [0,1] \mapsto e^{2\pi i t} \in S^2$. Then the map is continuous. Let us note that the basic open sets of S^1 are of the form either $B_1 = \{e^{2\pi i t} : t \in (a,b)\}$ for some $0 < a < b < 1 \in \mathbb{R}$ or $B_2 = \{e^{2\pi i t} : t \in [0,a) \cup (b,1]\}$. The preimage of $B_1 = (a,b)$ and the preimage of $B_2 = [0,a) \cup (b,1]$, both which are open in $[0,1]$. Thus the quotient of the interval $[0,1]$ obtained by identifying the end points is the circle.

21.20. Theorem. *Let X be a topological space Y carry the quotient topology obtained by $f\colon X \to Y$. Let Z be a topological space and $g\colon Y \to Z$ be a function. Then g is continuous if and only if $g \circ f$ is continuous.*

PROOF. If g is continuous, then $g \circ f$, being a composition of two continuous functions, is a continuous function. On the other hand suppose $g \circ f$ is continuous. Let $G \subset Z$ be open. Then $f^{-1}(g^{-1}(G))$ is open. Hence, $g^{-1}(G)$ is open, and hence, g is continuous. □

Next we give a construction of a quotient space that really justifies the name quotient.

Recall the Definition 5.9 as well as Definition 5.12 of quotient. Also, recall the construction of a quotient by means of a function in Theorem 5.11. The following theorem constructs a (topological) quotient of a topological space in the same way.

21.21. Theorem. *Let X, Y be two topological spaces and $f: X \to Y$ be an onto continuous function. As in Theorem 5.11, let $\sim_f$ be defined on X by $x \sim_f y$ if $f(x) = f(y)$, $(x, y \in X)$ and the function $\bar{f}: X/\sim_f \to Y$ be defined by $\bar{f}([x]) = f(x)$, $([x] \in X/\sim_f)$. Then $\bar{f}$ is a one to one correspondence between $X/\sim_f$ and Y.*

21.22. Theorem. *Let X, Y be two topological spaces and $f: X \to Y$ be an onto continuous function. Then the quotient top;ology $\mathscr{T}_q(q_f)$ on $X/\sim_f$ given by the quotient map $q_f: X \to X/\sim_f$ is the strongest topology such that each of the maps in the following diagram is continuous.*

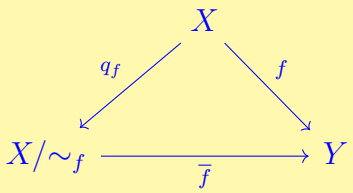


FIGURE 18. Quotient Space

PROOF. Clearly, $\mathscr{T}_q(q_f)$ is the strongest topology for which the map q_f is continuous. That is, q_f is the quotient map. Hence, by Theorem 21.22, $\overline{f}$ is continuous if and only if $f = \overline{f} \circ q_f$ is continuous. But f is continuous by the hypothesis. So, $\overline{f}$ is continuous. This completes the proof. □

21.23. Remark. As noted in Theorem 5.11, $\overline{f}$ is a continuous bijection. But it need not be a homeomorphism. The following example clears it.

21.24. Example. Consider $X = \{a, b, c\}$ with the topology $\mathscr{T}_X = \{\emptyset, X, \{a\}, \{a, b\}\}$, $Y = \{a, b, c\}$ with the topology $\mathscr{T}_Y = \{\emptyset, Y, \{a\}\}$ and $f: X \to Y$ the identity map. Clearly $X/\sim_f = X$ with the identity map. Hence the topology of $X/\sim_f$ is the same as that of X. But $X/\sim_f$ is not homeomorphic to Y.

21.25. Remark. The trick in the above example is to take two topologies $\mathscr{T}_Y < \mathscr{T}_X$ so that they are not homeomorphic. Since we are taking the identity map, it will be continuous and also the quotient will be homeomorphic to X.

21.26. Theorem. *Let X be a topological space and $A \subset X$. Denote by X/A, the quotient of X obtained by identifying the points of A. Let $f: X \to X/A$ be the quotient map.*

(1) If A is open, then f is open.
(2) If A is closed, then f is closed.

PROOF. Let $B \subset X$. If $\emptyset \neq B \subset A$, then $f^{-1}(f(B)) = A$. If $B \cap A = \emptyset$, then $f^{-1}(f(B)) = B$. If $B \cap A \neq \emptyset$, then $f^{-1}(f(B)) = B \cup A$.

☞(1) If $G \subset X$ is open, then as observed above, $f^{-1}(f(G))$ is A, G or $A \cup G$, each of which is open in X, and hence, $f(G)$ is open in X/A.

☞(2) If $F \subset X$ is closed, then as observed above, $f^{-1}(f(F))$ is A, F or $A \cup F$, each of which is closed in X, and hence, $f(F)$ is closed in X/A. □

Problems

151. Prove the Proposition 21.15.

152. Show that a circle is obtained by identifying two end points of a closed interval $[a, b]$.

153. Let $S = [0, 1] \times [0, 1]$. Identify $(0, y)$ with $(1, y)$ for every $y \in [0, 1]$. Show that the quotient space so obtained is homeomorphic to the cylinder.

154. Without using the Theorem 21.26, show that the quotient map defined in the Example 21.19 is closed but not open.

22. Product Topology

In this section, we define and investigate the basic concepts of the product spaces. Throughout this section, unless otherwise mentioned $\{(X_i, \mathscr{T}_i) : i \in I\}$ will be a family of topological spaces. We shall denote $\prod_{i\in I} X_i$ by X. Also for $j \in I$, $\pi_j \colon X \to X_j$ will be the j^{th} projection map.

22.1. Definition. The weak topology on X generated by all projection maps is called *the product topology* on X and is usually denoted by $\mathscr{T}_\pi$.

Clearly,

$$\mathscr{S}_\pi = \{\pi_i^{-1}(U_i) : U_i \in \mathscr{T}_i, i \in I\} \tag{22.1.1}$$

is a subbase and

$$\mathscr{B}_\pi = \{\cap_{k=1}^n \pi_{i_k}^{-1}(U_{i_k}) : U_{i_k} \in \mathscr{T}_{i_k} \text{ for every } i_k \in I, n \in \mathbb{N}\} \tag{22.1.2}$$

is a base for the product topology.

22.2. Theorem. *Let $\{X_i : i \in I\}$ be a family of topological spaces. For each $i \in I$, let $\mathscr{B}_i$ be a base for $\mathscr{T}_i$. Then*

$$\mathscr{B} = \{\cap_{k=1}^n \pi_{i_k}^{-1}(B_{i_k}) : B_{i_k} \in \mathscr{B}_{i_k} \text{ for every } i_k \in I, n \in \mathbb{N}\}$$

is a base for the product topology.

Proof. Let $U \in \mathscr{T}_\pi$ and $\mathbf{x} \in U$. Then there are $i_1, i_2, \ldots, i_n \in I$ and $U_{i_1} \in \mathscr{T}_{i_1}, U_{i_2} \in \mathscr{T}_{i_2}, \ldots, U_{i_n} \in \mathscr{T}_{i_n}$ such that $\mathbf{x} \in \cap_{k=1}^n \pi_{i_k}^{-1}(U_{i_k}) \subset U$. But then for every open sets $U_{i_k} \in \mathscr{T}_{i_k}$, there is $B_{i_k} \in \mathscr{B}_{i_k}$ such that $\mathbf{x}_{i_k} \in B_{i_k} \subset U_{i_k}$. Hence,

$$\mathbf{x} \in \cap_{k=1}^n \pi_{i_k}^{-1}(B_{i_k}) \subset \cap_{k=1}^n \pi_{i_k}^{-1}(U_{i_k}) \subset U.$$

This completes the proof. □

22.3. Corollary. *Let $\{X_i : i \in I\}$ be a family of topological spaces. For each $i \in I$, let $\mathscr{B}_i$ be a base for $\mathscr{T}_i$. Then*

$$\mathscr{S} = \{\pi_i^{-1}(B) : i \in I, B \in \mathscr{B}_i\}$$

is a subbase for the product topology.

Product of two ordered spaces need not be an order space. In order to make a distinction between an open interval (a, b) and the ordered pair (a, b) in a product space, we shall denote the ordered pair by $a \times b$.

22.4. Example. Consider $\mathbb{N}$ with the order topology. It is the discrete topology. Hence $\mathbb{N} \times \mathbb{N}$ with the product topology is discrete. Recall that the natural order on the product of two ordered sets is the *lexicographic order*. $\mathbb{N} \times \mathbb{N}$ with the order topology is not the discrete topology because any open set containing 2×1 will contain an interval of the form $(1 \times n, 2 \times k)$, and hence, is infinite. Thus $\{2 \times 1\}$ is not open.

22.5. Example. The order topology on $\mathbb{R}^2$ has the following as a basis.

$$\mathscr{B} = \{(a \times b, a, \times c) : a, b, c \in \mathbb{R}, b < c\}.$$

Thus for any open rectangle $(a, b) \times (c, d)$, we have

$$(a, b) \times (c, d) = \cup_{x \in (a,b)} \{x\} \times (c, d) = \cup_{x \in (a,b)} (x \times c, x \times d).$$

Thus the open rectangles are open in the order topology. But $(x \times c, x \times d)$ does not contain any open rectangle, and hence, it is not open in the usual topology on $\mathbb{R}^2$. Thus the usual topology on $\mathbb{R}^2$ is strictly weaker than the order topology.

22.6. Lemma. *Let X, Y be two ordered spaces. If Y does not have the minimum and maximum, then $X \times Y$ with the order topology is the product space $X_d \times Y$, where X_d denotes the discrete space.*

Proof. Since Y does not have the minimum as well as maximum, by the Proposition 15.1 (3) and (4), it follows that the order topology on Y is given by the base $\{(a, b) : a, b \in Y, a < b\}$. Thus the product topology on $X_d \times Y$ is given by $\{\{x\} \times (a, b) : x \in X, a, b \in Y, a < b\}$. Clearly, $\{x\} \times (a, b) = (x \times a, x \times b)$, proving that it is open in the order topology on $X \times Y$. On the other hand, for $x < y$ in X, $(x \times a, y \times b)$ is the union of $\{x\} \times (a, \infty)$, $\{y\} \times (-\infty, b)$ and for each $z \in (x, y)$, $\{z\} \times Y$, each of which is open in $X_d \times Y$. Thus $X \times Y$ with the order topology is the same as $X_d \times Y$. □

22.7. Theorem. *Projections are open.*

Proof. Let $S = \pi_j^{-1}(U_j) \in \mathscr{S}_\pi$ with $U_j \in \mathscr{T}_j$. By the Lemma 4.10 (5), $\pi_j(S) = U_j$, which is open in X_j. Let $S = \pi_i^{-1}(U_i)$ for some $i \neq j$. Then by the Lemma 4.10 (6), $\pi_j(S) = X_j$ if $U_i \neq \emptyset$ and $\pi_j(S) = \emptyset$, if $U_i = \emptyset$, which is also an open set. Now let

$$B = \cap_{k=1}^n S_k \in \mathscr{B}_\pi, \tag{22.7.1}$$

with $S_k \in \mathscr{S}$ for every $1 \leq k \leq n$.

In the equation, (22.7.1), assume that for some $i \in I$, $1 \leq k < m \leq n$, $S_k = \pi_i^{-1}(G)$ and $S_m = \pi_i^{-1}(H)$, where $G, H \in \mathscr{T}_i$ are open. Then $S_k \cap S_m = \pi_i^{-1}(G) \cap \pi_i^{-1}(H) = \pi_i^{-1}(G \cap H)$. Replacing the $S_k \cap S_m$ by $\pi_i^{-1}(G \cap H)$, the repetition is removed. Carrying out this replacement finite number of time, we can assume without loss of generality that

$$B = \cap_{k=1}^n S_k = \pi_{i_1}^{-1}(U_{i_1}) \cap \pi_{i_2}^{-1}(U_{i_2}) \cap \cdots \cap \pi_{i_n}^{-1}(U_{i_n}),$$

with $S_k = \pi_{i_1}^{-1}(U_{i_1}) \in \mathscr{S}$ for every $1 \leq k \leq n$ and $i_k \neq i_m$ if $1 \leq k < m \leq n$. Now applying the Lemma 4.10 (3) to each S_k and simplifying, we get

$$B = \cap_{k=1}^n S_k = \cap_{k=1}^n \pi_{i_k}^{-1}(U_{i_k}) = \prod_{i \in I} C_i,$$

where $C_i = U_i$ if $i = i_k$ for some $1 \leq k \leq n$ and $C_i = X_i$ otherwise. Thus $\pi_j(B) = C_j$, which is open in X_j.

Finally, for arbitrary $V \in \mathscr{T}_\pi$, with $V = \cup_{\alpha \in \Lambda} B_\alpha$, and $B_\alpha \in \mathscr{B}_p$ for all $\alpha \in \Lambda$,

$$\pi_j(V) = \pi_j(\cup_{\alpha \in \Lambda} B_\alpha) = \cup_{\alpha \in \Lambda} \pi_j(B_\alpha),$$

which is open in X_j as $\pi_j(B_\alpha)$ is open in X_j for all $\alpha \in \Lambda$. □

The projections are not closed in general. As seen in the Example 19.3, the projection $\pi_1 \colon \mathbb{R}^2 \to \mathbb{R}$, $\pi_1(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_1$, $((\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^2)$, is not closed, because $A = \{(\frac{1}{n}, n) : n \in \mathbb{N}\}$ is closed in $\mathbb{R}^2$. But $\pi_1(A) = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed.

22.8. Lemma. *Let $f \colon (0, \infty) \to (0, \infty)$ be a continuous function. Then the sets*

$$H_1 = \{(\mathbf{x}_1, \mathbf{x}_2) \in (0, \infty) \times (0, \infty) : f(\mathbf{x}_1) < \mathbf{x}_2\}$$

and

$$H_2 = \{(\mathbf{x}_1, \mathbf{x}_2) \in (0, \infty) \times (0, \infty) : f(\mathbf{x}_1) > \mathbf{x}_2\}$$

are open in $(0, \infty) \times (0, \infty)$.

Proof. We prove that H_1 is open and similarly we can prove that H_2 is open. Let $(\mathbf{a}_1, \mathbf{a}_2) \in H_1$. Hence $\mathbf{a}_2 > f(\mathbf{a}_1)$. Let $r = \frac{\mathbf{a}_2 - f(\mathbf{a}_2)}{2}$. Then there exists $\delta > 0$ such that $f((\mathbf{a}_1 - \delta, \mathbf{a}_1 + \delta)) \subset (f(\mathbf{a}_1) - r, f(\mathbf{a}_1) + r)$. Let $G = (\mathbf{a}_1 - \delta, \mathbf{a}_1 + \delta) \times (\mathbf{a}_2 - r, \mathbf{a}_2 + r)$. Taking $(\mathbf{x}_1, \mathbf{x}_2) \in G$, we see that $f(\mathbf{a}_1) + r = f(\mathbf{a}_1) + \frac{\mathbf{a}_2 - f(\mathbf{a}_1)}{2} = \frac{2f(\mathbf{a}_1) + \mathbf{a}_2 - f(\mathbf{a}_1)}{2} = \frac{\mathbf{a}_2 + f(\mathbf{a}_1)}{2}$. Also $\mathbf{a}_2 - r = \mathbf{a}_2 - \frac{\mathbf{a}_2 - f(\mathbf{a}_1)}{2} = \frac{2\mathbf{a}_2 - \mathbf{a}_2 + f(\mathbf{a}_1)}{2} = \frac{\mathbf{a}_2 + f(\mathbf{a}_1)}{2}$. Now let $(\mathbf{x}_1, \mathbf{x}_2) \in G$. Then $\mathbf{x}_1 \in (\mathbf{a}_1 - \delta, \mathbf{a}_1 + \delta)$. So, $f(\mathbf{x}_1) < f(\mathbf{a}_1) + r = \mathbf{a}_2 - r < \mathbf{x}_2$. Thus $(\mathbf{x}_1, \mathbf{x}_2) \in H_1$. Hence $G \subset H_1$. Thus H_1 is open in $(0, \infty) \times (0, \infty)$. □

Since the product topology is the weak topology determined by the projections, the following is at once.

22.9. Theorem. *Let Y be a topological space and $f \colon Y \to X$ be a function. Then f is continuous if and only if $\pi_i \circ f$ is continuous for every projection π_i.*

22.10. Corollary. *Let $\{X_i : i \in I\}$ be a family of topological spaces, $X = \prod_{i\in I} X_i$ and Y be a topological space. Suppose for each $i \in I$, there is a continuous map $f_i\colon Y \to X_i$. Then the lifting f of $\{f_i : i \in I\}$ is continuous.*

Proof. For every $i \in I$, $\pi_i \circ f = f_i$ is continuous. So, by the Theorem 22.9 f is continuous. □

22.11. Theorem. *Let $\{X_i : i \in I\}$ be a family of topological spaces and $X = \prod_{i\in I} X_i$. Let Y be a topological space and $f_i\colon Y \to X_i$, $(i \in I)$, be mappings. Then the lifting f of $\{f_i : i \in I\}$ is embedding if and only if*

(1) $\{f_i : i \in I\}$ separates the points of Y and
(2) Y carries the weak topology given by $\{f_i : i \in I\}$.

Proof. Suppose f is embedding. So, f is one-one, and hence, by the Theorem 4.15, $\{f_i : i \in I\}$ separates the points of Y. Also, since f is continuous, so are f_i for all $i \in I$. Hence $f_i^{-1}(U_i)$ are open for every i and every open U_i in X_i. On the other hand, let $U \subset Y$ be an open set in Y and $x \in U$. Since f is embedding, $f(U)$ is open in $f(Y)$. So there are $i_1, i_2, \ldots, i_n \in I$ and open sets $U_{i_1}, U_{i_2}, \ldots, U_{i_n}$ in $X_{i_1}, X_{i_2}, \ldots, X_{i_n}$ respectively, such that $f(x) \in \cap_{k=1}^n(\pi_{i_k}^{-1}(U_{i_k}) \cap f(Y)) \subset f(U)$. But then

$$x \in f^{-1}(\cap_{k=1}^n(\pi_{i_k}^{-1}(U_{i_k}))) = \cap_{k=1}^n(f^{-1}(\pi_{i_k}^{-1}(U_{i_k}))) = \cap_{k=1}^n(f_{i_k}^{-1}(U_{i_k})) \subset U.$$

Thus Y has the weak topology generated by $\{f_i : i \in I\}$.

Conversely, suppose Y has the weak topology given by $\{f_i : i \in I\}$. Clearly, by the Corollary 22.10, f is continuous. Let $U \subset Y$ be open. So, $U = \cap_{k=1}^n f_{i_k}^{-1}(U_{i_k})$ for some open sets $U_{i_1}, U_{i_2}, \ldots, U_{i_n}$ in $X_{i_1}, Xi_2, \ldots, X_{i_n}$ respectively. But then $f(U) = f(\cap_{k=1}^n f_{i_k}^{-1}(U_{i_k})) = f(\cap_{k=1}^n(f^{-1}(\pi_{i_k}^{-1}(U_{i_k})))) = ff^{-1}(\cap_{k=1}^n(\pi_{i_k}^{-1}(U_{i_k}))) = (\cap_{k=1}^n(\pi_{i_k}^{-1}(U_{i_k}))) \cap f(Y)$. Thus f is a homeomorphism. □

22.12. Definition. Let $\mathbf{a} = (\mathbf{a}_i) \in \prod_{i\in I} X_i$. For $j \in I$. We define

$$X(\mathbf{a}, j) = \{\mathbf{x} = (\mathbf{x}_i) : \mathbf{x}_i = \mathbf{a}_i, \text{ for } i \neq j\}.$$

22.13. Theorem. *Let $\mathbf{a} = (\mathbf{a}_i) \in \prod_{i\in I} X_i$. For $j \in I$, $X(\mathbf{a}, j) \cong X_j$.*

Proof as a Corollary to the Theorem 22.11. For every $i \neq j$, let $f_i\colon X_j \to X_i$ be the constant map defined by $f_i(x) = \mathbf{a}_i$, $(x \in X)$. Also, for $i = j$, let f_j be the identity map. Then $\{f_i : i \in I\}$ separates the points of X_j and X_j carries the weak topology given by $\{f_i : i \in I\}$. Thus, X_j is homeomorphic to $f(X_j) = X(\mathbf{a}, j)$, where f is the lifting of $\{f_i : i \in I\}$. □

Traditional Proof. Clearly, $\pi_{j|_{X(\mathbf{a},j)}}\colon X(\mathbf{a}, j) \to X_j$ is continuous. Let $\mathbf{x}, \mathbf{y} \in X(\mathbf{a}, j)$ be two distinct points. But then the only possible distinction in the

coordinates of $\mathbf{x}$ and $\mathbf{y}$ is in their j^{th} coordinates. Thus $\pi_{j|_{X(\mathbf{a},j)}}(\mathbf{x}) = \mathbf{x}_j \neq \mathbf{y}_j = \pi_{j|_{X(\mathbf{a},j)}}(\mathbf{y})$. Thus $\pi_{j|_{X(\mathbf{a},j)}}$ is one-one. Also for $z \in X_j$, define

$$\mathbf{x}_i = \begin{cases} \mathbf{a}_i, & \text{if } i \neq j; \\ z, & \text{if } i = j. \end{cases}$$

Then $\mathbf{x} \in X(\mathbf{a}, j)$ and $\pi_{j|_{X(\mathbf{a},j)}}(x) = z$. Thus $\pi_{j|_{X(\mathbf{a},j)}}$ is onto.

Finally, let $\pi_i^{-1}(U_i) \in \mathscr{S}_\pi$. Assume first that $i \neq j$ If $\mathbf{a}_i \notin U_i$, then $\pi_i^{-1}(U_i) \cap X(\mathbf{a}, j) = \emptyset$, and hence, $\pi_{j|_{X(\mathbf{a},j)}}(\pi_i^{-1}(U_i) \cap X(\mathbf{a}, j)) = \emptyset$. If $\mathbf{a}_i \in U_i$, then $\pi_{j|_{X(\mathbf{a},j)}}(\pi_i^{-1}(U_i) \cap X(\mathbf{a}, j)) = X(\mathbf{a}, j)$, and hence, $\pi_{j|_{X(\mathbf{a},j)}}(\pi_i^{-1}(U_i) \cap X(\mathbf{a}, j)) = X_j$.

Let $i = j$. Then $\pi_j((\pi_{j|_{X(\mathbf{a},j)}})^{-1}(U_j)) = U_j \Rightarrow \pi_{j|_{X(\mathbf{a},j)}}(\pi_i^{-1}(U_i) \cap X(\mathbf{a}, j)) = U_j$.

Thus, we conclude that $\pi_{j|_{X(\mathbf{a},j)}} : X(\mathbf{a}, j) \to X_j$ is a homeomorphism. □

22.14. Corollary. *X_j is homeomorphic to a subspace of $\prod_{i\in I} X_i$.*

22.15. Definition. Let $\{X_i : i \in I\}$ be a family of topological spaces, X be a topological space and for every $i \in I$, $f_i \colon X \to X_i$ be a continuous function. We say that $\{f_i : i \in I\}$ *separates points from closed sets* if whenever E is a closed set in X and $x \in X \setminus E$, there exists $i \in I$ such that $f_i(x) \notin \overline{f(E)}$.

The following is an exercise in [**Wil70**, Exercise 8B]. We are interested in [**Wil70**, Corollary 8.15].

22.16. Theorem. *Let $\{X_i : i \in I\}$ be a family of topological spaces, X be a topological space and for every $i \in I$, $f_i \colon X \to X_i$ be a continuous function. Then $\{f_i : i \in I\}$ separates points from closed sets if and only if $\{f_i^{-1}(V) : i \in I, V$ is open in $X_i\}$ forms a base for the topology of X.*

Proof. Assume that $\{f_i : i \in I\}$ separates points from closed sets. Let U be an open set in X and $x \in U$. Then $E = X \setminus U$ is a closed set and $x \notin E$. So, there is $i \in I$ such that $f_i(x) \notin \overline{f_i(E)}$. So, there is an open set V in X_i such that $f(x) \in V \subset X_i \setminus \overline{f_i(E)}$. Since f_i is continuous and since E is closed, $f_i(E) = f_i(\overline{E}) \subset \overline{f_i(E)}$. Hence $X_i \setminus \overline{f_i(E)} \subset X_i \setminus f_i(E)$. Thus $f(x) \in V \subset X_i \setminus f_i(E)$. Hence $x \in f^{-1}(V) \subset X \setminus E = U$. Thus $\{f_i^{-1}(V) : i \in I, V$ is open in $X_i\}$ forms a base for the topology of X.

Conversely, suppose $\{f_i^{-1}(V) : i \in I, V$ is open in $X_i\}$ forms a base for the topology of X. Let E be a closed subset of Y and $x \notin E$. So $U = Y \setminus E$ is open and $x \in U$. Hence there is $i \in I$ and an open set $V \subset X_i$ such that $x \in f_i^{-1}(V) \subset U$ and so, $f_i^{-1}(V) \cap E = \emptyset$. But then $f_i(x) \in V$ and $V \cap f_i(E) = \emptyset$. Since V is open, $V \cap \overline{f_i(E)} = \emptyset$. Hence $f_i(x) \notin \overline{f_i(E)}$. □

The following is at once now.

22.17. Corollary. *Let $\{X_i : i \in I\}$ be a family of topological spaces, X be a topological space and for every $i \in I$, $f_i \colon X \to X_i$ be a continuous function. If $\{f_i : i \in I\}$ separates points from closed sets, then X carries the weak topology given by $\{f_i : i \in I\}$.*

Problems

155. Let $X = Y = \mathbb{N}$, and $Z = \{a, b\}$, all the three with the discrete topologies. Show that $X \times Y$ and $X \times Z$ are homeomorphic. However, the cancellation does not hold, that is, Y is not homeomorphic to Z.

156. Let $\{X_i : i \in I\}$ be a family of topological spaces and $X = \prod_{i\in I} X_i$. Show that $\mathscr{B}_b = \{\prod_{i\in I} U_i : U_i \text{ is open in } X_i \text{ for every } i \in I\}$ is a base for a topology, say $\mathscr{T}_b$. Also show that the topology coincides with the product topology if and only if all but finitely many X_i are indiscrete. The topology $\mathscr{T}_b$ is called the *box topology*.

157. Let $\{X_i : i \in I\}$ be a family of topological spaces and $U \subset \prod_{i\in I} X_i$ be open. Show that $\{i \in I : \pi_i(U) \neq X_i\}$ is finite.

158. For a product space $X = \prod_{i\in I} X_i$ and $\mathbf{a} = (\mathbf{a}_i) \in X$, show that $A = \{\mathbf{x} \in X : \mathbf{x}_i = \mathbf{a}_i \text{ for all but finitely many } i \in I\}$ is dense in X.

159. Let $\{X_i : i \in I\}$ be a family of topological spaces and $Y_i \subset X_i$ for every $i \in I$. Show that the product topology on $Y = \prod_{i\in I} Y_i$ coincides with the subspace topology inherited from the product space $X = \prod_{i\in I} X_i$. That is, show that product of subspaces is a subspace of the product space.

160. Let $\{X_i : i \in I\}$ be a family of topological spaces. Prove the following.

(1) Product of two spaces is discrete if and only if each of them is discrete.
(2) Product of topological spaces is indiscrete if and only if each of the factor spaces is indiscrete.
(3) A countable product of discrete spaces, each with at least two points, cannot be discrete.
(4) Let $I = J \cup K$ with $J \cap K = \emptyset$. Show that the product space $\prod_{i\in I} X_i$ is homeomorphic to $\prod_{i\in J} X_i \times \prod_{i\in K} X_i$.
(5) Let $f \colon I \to I$ be a bijective function. Show that

$$\prod_{i\in I} X_i \cong \prod_{i\in I} X_{f(i)}.$$

161. Show that $A = \{(\mathbf{x}, \frac{1}{\mathbf{x}}) : \mathbf{x} > 0\}$ is closed in $\mathbb{R} \times \mathbb{R}$ but $\pi_1(A)$ is not closed in $\mathbb{R}$.

162. Let $\mathbb{R}^2_\ell = \mathbb{R}_\ell \times \mathbb{R}_\ell$. Show that

(1) $Y = \{(\mathbf{x}, -\mathbf{x}) : \mathbf{x} \in \mathbb{R}\}$ is discrete.
(2) Let $Z = \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbb{R}\}$. Show that the restrictions of π_1, π_2 to B are hoeomorphisms.

163. Consider $\mathbb{R} \times \mathbb{R}_d$, where $\mathbb{R}_d$ is the $\mathbb{R}$ with the discrete topology. For $m \in \mathbb{R} \setminus \{0\}$, show that the projection map π_1 restricted to $Y = \{(x, mx) : x \in \mathbb{R}\}$ is not open.

164. Let $\mathbb{R}^2_{\ell u} = \mathbb{R}_\ell \times \mathbb{R}_u$. Show that

(1) $Y = \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \mathbb{R}\}$ is discrete.
(2) Let $Z = \{(\mathbf{x}, -\mathbf{x}) : \mathbf{x} \in \mathbb{R}\}$. Show that restriction of π_1, π_2 to B are hoeomorphisms.

165. Let $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ be two families of topological spaces. Suppose for every $i \in I$, $f_i \colon X_i \to Y_i$ is a homeomorphism. Also, define $F \colon \prod X_i \to \prod Y_i$, defined by $F(\mathbf{x})_i = f_i(\mathbf{x}_i)$, $(\mathbf{x} \in \prod X_i, i \in I)$. Show that F is a homeomorphism.

166. Let $\{X_i : i \in I\}$ and $X = \prod_{i\in I} X_i$. For $A_i \subset X_i$, prove the following.

(1) $\prod_{i\in I} \overline{A_i} = \overline{\prod_{i\in I} A_i}$.
(2) If all but finitely many X_i is indiscrete, then $\prod_{i\in I}(A_i^\circ) = (\prod_{i\in I} A_i)^\circ$.
(3) If infinitely many X_i are not indiscrete, then there are $A_i \subset X_i$ such that $\prod_{i\in I}(A_i^\circ) \neq (\prod_{i\in I} A_i)^\circ$.

167. Let X be a set and $\{\mathscr{T}_i : i \in I\}$ be a family of topologies on X. Let X_i denote the topological space $(X, \mathscr{T}_i)$. Show that the weak topology on X given by $\{\iota_i \colon X \to X_i : i \in I\}$ is homeomorphic to $\Delta = \{\mathbf{x} \in \prod_{i\in I} X_i : \mathbf{x}_i = \mathbf{x}_j \text{ for every } i, j \in I\}$

23. Coproduct

Dual to the product, there is the concept of coproduct, better known as disjoint union. In this section, we define and investigate the basic concepts of the coproduct spaces. Throughout this section, unless otherwise mentioned $\{(X_i, \mathscr{T}_i) : i \in i\}$ will be a family of topological spaces. We shall denote $\coprod_{i\in I} X_i$ by X. Also for $j \in I$, $\iota_j : x \in X_j \mapsto (x, j) \in X$ will be the injection maps.

23.1. Definition. The strongest topology on X generated by all injection maps is called *the coproduct topology* on X and is denoted by $\mathscr{T}_{cp}$.

23.2. Remark.

(1) Since the topology on X is the strongest topology such that each injection is continuous, by definition injections are continuous.
(2) $G \subset X$ is open if and only if $G_i = \{x \in X_i : (x, i) \in G\}$ is open in X_i for every $i \in I$ if and only if $G = \coprod_{i\in I} G_i$, where G_i is open in X_i for each i.
(3) $\iota_i(X_i) = X_i^*$ and in fact, each ι_i is an embedding, giving a homeomorphism $\iota_i \colon X_i \to X_i^*$.

23.3. Theorem. *ι_j are open as well as closed maps.*

Proof. Let $U \subset X_j$ be open. Then $\iota_j^{-1}(\iota_j(U)) = U$ is open in X_j and $\iota_i^{-1}(\iota_j(U)) = \emptyset$ is open in X_i, $(i \neq j)$. So, $\iota_j(U)$ is open in X.

Let $F \subset X_j$ be closed in X_j. Then $X \setminus \iota_j(E) = \coprod_{i\in I} G_i$, where $G_i = X_i$ if $i \neq j$ and $G_j = X_j \setminus E$. Thus $X \setminus \iota_j(E)$ is open. This completes the proof. □

23.4. Corollary. *For every $j \in I$, X_j^* is a clopen subset of $\coprod_{i\in I} X_i$.*

23.5. Theorem. *For each $i \in I$, let $\mathscr{B}_i$ be a base on X_i. Then*

$$\mathscr{B} = \{B_i \coprod \emptyset : B_i \in \mathscr{B}_i, i \in I\}$$

is a base for X.

Proof. Let $(x, j) \in X$. Then there is $B \in X_j$ such that $x \in B$. Then $(x, j) \in B \coprod \emptyset \in \mathscr{B}$.

Let $B_1 = B_{j,1} \coprod \emptyset, B_2 = B_{k,2} \coprod \emptyset \in \mathscr{B}$ and $(x, m) \in B_1 \cap B_2$. Then clearly, $j = k = m$ and so, there is $B \in \mathscr{B}_j$ such that $x \in B \subset B_{j,1} \cap B_{j,2}$. Taking $B_3 = B \coprod \emptyset$, we have $(x, m) \in B_3 \subset B_1 \cap B_2$. This proves the result. □

23.6. Theorem. *Let Y be a topological space and $f : X = \coprod_{i\in I} X_i \to Y$ be a map. Then f is continuous if and only if $f \circ \iota_i$ is continuous for every $i \in I$.*

Proof. If f is continuous, then $f \circ \iota_i$ is continuous for every i. Conversely, assume that $f \circ \iota_i$ is continuous for every i. Let $U \subset Y$ be open and $(x, j) \in f^{-1}(U)$. So, $\iota_j^{-1}(f^{-1}(U)) = (f \circ \iota_j)^{-1}(U)$ is open in X_j. So, $(x, j) \in (f \circ \iota_j)^{-1}(U) \coprod \emptyset \subset f^{-1}(U)$. This completes the proof. □

Problems

168. Let $\{X_i : i \in I\}, \{Y_i : i \in I\}$ be families of topological spaces and $X = \coprod_{i\in I} X_i$ and $Y = \coprod_{i\in I} Y_i$. For each $i \in I$, let $f_i : X_i \to Y_i$ be a map. Define $f : X \to Y$ by $f(x, i) = (f_i(x), i)$, $(x, i) \in X$. Show that f is continuous if and only if each f_i is continuous.

169. Let X be a topological space and $\{U_i : i \in I\}$ be a family of open subsets of X such that $X = \cup_{i\in I} U_i$ and $U_i \cap U_j = \emptyset$ if $i \neq j$. Then show that $X \cong \coprod_{i\in I} U_i$.

24. Inverse Limit of Topological Spaces

24.1. Definition. Let I be a directed set and for every $i \in I$, let $(X_i, \mathscr{T}_i)$ be a topological space. For every $i, j \in I$ with $i \leq j$, let there be a continuous mapping $\pi_{j,i} : X_j \to X_i$ such that

(1) $\pi_{i,i} = \iota_i$, the identity map of X_i, for every $i \in I$, and
(2) the diagram in Figure 19 commutes whenever $i \leq j \leq k$. That is $\pi_{k,i} = \pi_{j,i} \circ \pi_{k,j}$.

Then $(\{X_i : i \in I\}, \{\pi_{j,i} : i \leq j\})$ is called an *inverse system* or *projective system* of topological spaces.

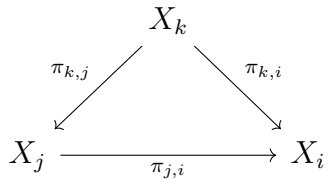


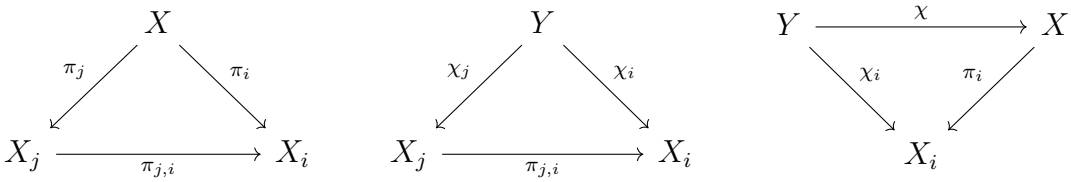
Figure 19. Inverse System

In this section, we fix the inverse system $(\{X_i : i \in I\}, \{\pi_{j,i} : i \leq j\})$ and denote it by $(\{X_i\}, \{\pi_{j,i}\})$.

24.2. Definition. Let $(\{X_i\}, \{\pi_{j,i}\})$ be an iverse system. A topological space X is said to be an *inverse limit* or a *projective limit* of $(\{X_i\}, \{\pi_{j,i}\})$, denoted by $\varprojlim_{i\in I} X_i$, if the following hold.

(1) For each $i \in I$, there is a continuous function $\pi_i \colon X \to X_i$ such that for every $i \leq j$, the diagram in Figure 20(a) commutes.

(2) If there is a topological space Y, and for every $i \in I$, a continuous function $\chi_i \colon Y \to X_i$ such that for every $i \leq j$, the diagram in Figure 20(b) commutes, then there is a unique continuous function $\chi \colon Y \to X$ such that for every $i \in I$, the diagram in Figure 20(c) commutes.



(a) The object X (b) Other Space (c) Attracting Y

FIGURE 20. Inverse Limit

The following theorem proves the existence of inverse limit.

24.3. Theorem. *Let $(\{X_i\}, \{\pi_{j,i}\})$ be an inverse system of topological spaces. Define*

$$X = \{\mathbf{x} = (\mathbf{x}_i) \in \prod_{i\in I} X_i : \pi_{j,i}(\mathbf{x}_j) = \mathbf{x}_i \text{ whenever } i \leq j\}. \tag{24.3.1}$$

Then $X = \varprojlim_{i\in I} X_i$.

PROOF. Restricting the projections to X, it is clear from the definition of X in (24.3.1) that for each $\mathbf{x} \in X$, the diagram in Figure 20(a) commutes. Now let Y be a topological space with a family $\{\chi_i \colon Y \to X_i : i \in I\}$ such that the diagram in Figure 20(a) commutes. But then by Theorem 4.12, there is unique map $\chi \colon Y \to \prod_{i\in I} X_i$ such that $\chi_i = \pi_i \circ \chi$ for every $i \in I$. Let $\chi(y) = \mathbf{x} \in \chi(Y)$ and $i \leq j$. Then

$$\begin{aligned}
\pi_{j,i}(\pi_j(\mathbf{x})) &= \pi_{j,i}(\pi_j(\chi(y))) \\
&= \pi_{j,i}(\chi(y)_j) \\
&= \pi_{j,i}(\chi_j(y)) \\
&= \chi_i(y) \\
&= \pi_i(\chi(y)) \\
&= \pi_i(\mathbf{x}).
\end{aligned}$$

Thus $\mathbf{x} \in X$. That is, $\chi(Y) \subset X$. Thus $\chi \colon Y \to X$. The uniqueness is at once. □

CHAPTER 5

Metric Spaces

The roots of topological spaces lie in the metric spaces and roots of metric spaces lie in the real line with the usual distance.

25. Metric Topology

25.1. Definition. Let X be a nonempty set. A *metric on X* is a function $d\colon X \times X \to \mathbb{R}$ satisfying the following.

(1) $d(x, y) \geq 0$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$. *(positive definiteness)*
(2) $d(x, y) = d(y, x)$, for all $x, y \in X$. *(symmetry)*
(3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$. *(triangle inequality)*

In this case, the set X together with the metric d is called a *metric space*. It is denoted by (X, d) or X.

25.2. Example. Let X be a nonempty set and $d\colon X \times X \to R$ be defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 1, & \text{if } x \neq y. \end{cases}$$

Then it is easily seen that d is a metric on X. The metric d in this case, is called the *discrete metric*.

Positive definiteness and symmetry are at once. For triangle inequality, let $x, y, z \in X$. If $x = y$, then $d(x, y) = 0 \leq d(x, z) + d(z, y)$. If $x \neq y$, then z cannot be equal to x and y both at the same time, say $x \neq z$. Consequently, $d(x, y) = 1 = d(x, z) \leq d(x, z) + d(z, y)$. Thus d is a metric on X.

Before we proceed further, we recall certain algebraic results.

25.3. Lemma. *For $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.*

Proof.

$$|x + y| = \max\{x + y, (-x) + (-y)\}$$

$$\begin{aligned}
&\leq \max\{|x| + y, |x| + (-y)\} && \text{(because } x \leq |x|, -x \leq |x|\text{)}\\
&\leq \max\{|x| + |y|, |x| + |y|\} && \text{(because } y \leq |y|, -y \leq |y|\text{)}\\
&= |x| + |y|. && \square
\end{aligned}$$

We now head towards the Hölder's inequality on $\mathbb{C}^n$. Hölder's inequality is used to prove the Minkowski's inequality, which, in turn, will be useful in proving the triangle inequality for various metrics on $\mathbb{R}^n$ and $\mathbb{C}^n$. In fact, the following inequalities hold for infinite sequences whenever the concerned series and the supremum define numbers.

25.4. Lemma. (Hölder's inequality) *Let* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n \in \mathbb{C}$ *and* $1 \leq p, q < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\left|\sum_{i=1}^{n} \mathbf{x}_i \mathbf{y}_i\right| \leq \left(\sum_{i=1}^{n} |\mathbf{x}_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |\mathbf{y}_i|^q\right)^{\frac{1}{q}}.$$

Also,

$$\left|\sum_{i=1}^{n} \mathbf{x}_i \mathbf{y}_i\right| \leq \left(\sum_{i=1}^{n} |\mathbf{x}_i|\right) \max_{1\leq i\leq n} |\mathbf{y}_i|.$$

25.5. Lemma. (Minkowski's inequality) *Let* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n \in \mathbb{C}$ *and* $1 \leq p < \infty$. *Then*

$$\left(\sum_{i=1}^{n} |\mathbf{x}_i + \mathbf{y}_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} |\mathbf{x}_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |\mathbf{y}_i|^p\right)^{\frac{1}{p}}.$$

Also,

$$\max_{1\leq i\leq n} |\mathbf{x}_i + \mathbf{y}_i| \leq \max_{1\leq i\leq n} |\mathbf{x}_i| + \max_{1\leq i\leq n} |\mathbf{y}_i|.$$

25.6. Example. Let $X = \mathbb{R}$. Define for $x, y \in \mathbb{R}$, $d(x, y) = |x - y|$. Then d is a metric on $\mathbb{R}$ called the *usual metric* on $\mathbb{R}$.

In this case also, the positive definiteness and the symmetry are at once. To prove the triangle inequality, let $x, y, z \in X$. Then we invoke the Lemma 25.3 in the following.

$$d(x, y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y| = d(x, z) + d(z, y).$$

25.7. Example. For $n \in \mathbb{N}$ and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n), y = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n) \in \mathbb{R}^n$, define

$$d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| = \sqrt{|\mathbf{x}_1 - \mathbf{y}_1|^2 + |\mathbf{x}_2 - \mathbf{y}_2|^2 + \cdots + |\mathbf{x}_n - \mathbf{y}_n|^2}.$$

$$d_p(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_p = \left(|\mathbf{x}_1 - \mathbf{y}_1|^p + |\mathbf{x}_2 - \mathbf{y}_2|^p + \cdots + |\mathbf{x}_n - \mathbf{y}_n|^p\right)^{\frac{1}{p}}.$$

$$d_\infty(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_\infty = \max_{1\leq i\leq n} |\mathbf{x}_i - \mathbf{y}_i|.$$

Then d, d_p, d_∞, $(1 \le p < \infty)$ are metrics on $\mathbb{R}^n$. The metric d is the same as the d_2 and is called the *usual* or *Euclidean* metric on $\mathbb{R}^n$.

Indeed the positive definiteness and symmetry are at once. However, the triangle inequality is involved and can be proved using Minkowski's inequality stated in the Lemma 25.5

25.8. Remark.

(1) In the above example, one can replace $\mathbb{R}^n$ by $\mathbb{C}^n$ without changing the formulae of the metrics. Still they will be metrics.

(2) Each metric d_p, $(1 \le p \le \infty)$, has its analogues for infinite sequences. For $1 \le p < \infty$, let

$$\ell^p = \{\mathbf{x} = (\mathbf{x}_n)_{n=1}^\infty : \sum_{n=1}^\infty |\mathbf{x}_n|^p < \infty\}$$

and

$$\ell^\infty = \{\mathbf{x} = (\mathbf{x}_n)_{n=1}^\infty : \sup_{1\le n\le\infty} |\mathbf{x}_n| < \infty\}.$$

For $\mathbf{x}, \mathbf{y} \in \ell^p$, $(1 \le p < \infty)$, define $d_p(\mathbf{x}, \mathbf{y}) = (\sum_{n=1}^\infty |\mathbf{x}_n - \mathbf{y}_n|^p)^{\frac{1}{p}}$. Also, for $\mathbf{x}, \mathbf{y} \in \ell^\infty$, define $d_\infty(\mathbf{x}, \mathbf{y}) = \sup_{1\le n<\infty} |\mathbf{x}_n - \mathbf{y}_n|$. Then (ℓ^p, d_p) is a metric space for $1 \le p \le \infty$.

25.9. Example. Let (X, d) be a metric space and $r > 0$. Define

$$d_r(x, y) = rd(x, y), (x, y \in X).$$

Then d_r is a metric on X.

25.10. Example. Let (X, d) be a metric space. Define

$$\bar{d}(x, y) = \min\{d(x, y), 1\}, (x, y \in X) \tag{25.10.1}$$

and

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}, (x, y \in X). \tag{25.10.2}$$

Then $\bar{d}$ and ρ are metrics on X. The positive definiteness and the symmetry of ρ are consequences of those of d. Now for $x, y, z \in X$,

$$\begin{aligned}
\rho(x, y) &= \frac{d(x, y)}{1 + d(x, y)} = \frac{1 + d(x, y) - 1}{1 + d(x, y)} = 1 - \frac{1}{1 + d(x, y)} \\
&\le 1 - \frac{1}{1 + d(x, z) + d(z, y)} = \frac{1 + d(x, z) + d(z, y) - 1}{1 + d(x, z) + d(z, y)} \\
&= \frac{d(x, z)}{1 + d(x, z) + d(z, y)} + \frac{d(z, y)}{1 + d(x, z) + d(z, y)} \\
&\le \frac{d(x, z)}{1 + d(x, z)} + \frac{d(z, y)}{1 + d(z, y)} = \rho(x, z) + \rho(z, y).
\end{aligned}$$

25.11. Example. Let (X, d) be a metric space and $a \in X$ be fixed. Then $d\colon X \times X \to \mathbb{R}$, defined by

$$d_T(x, y) = \begin{cases} 0, & \text{if } x = y; \\ d(x, a) + d(a, y), & \text{if } x \ne y, \end{cases}$$

is a metric on X and is known the *telephone metric*.

25.12. Definition. Let (X, d) be a metric space, $x \in X$ and $\varepsilon > 0$. The set

$$B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$$

is called the *open ε-ball centred at x* or *open ball centred at x and of radius ε*. Many authors also call this set an *open sphere* instead of an open ball and denote it by $S_\varepsilon(x)$.

25.13. Definition. Let (X, d) be a metric space, $x \in X$ and $\varepsilon > 0$. The set

$$B_d[x, \varepsilon] = \{y \in X : d(x, y) \leq \varepsilon\}$$

is called the *closed ε-ball centred at x* or the *closed ball centred at x and of radius ε*. Many authors also call this set a *closed sphere* instead of a closed ball and is denoted by $S_\varepsilon[x]$.

Note: For brevity, instead of writing $B_d(x, \varepsilon)$ we shall write $B(x, \varepsilon)$.

25.14. Proposition. *Let (X, d) be a metric space, $x \in X$ and $r > 0$. Then $B[x, r]$ is closed.*

Proof. Let $y \notin B[x, r]$. Then $d(x, y) > r$. Let $t = d(x, y) - r$. Suppose, if possible, that $z \in B[x, r] \cap B(y, t)$. Then $d(x, y) \leq d(x, z) + d(z, y) < r + d(x, y) - r = d(x, y)$ a contradiction. Thus $B(y, t)$ does not contain any point of $B[x, r]$, and so, y is not a limit point of $B[x, r]$. That means, $B[x, r]$ contains all its limit points, and hence, it is closed. □

25.15. Corollary. *Let (X, d) be a metric space, $x \in X$ and $r > 0$. Then $\overline{B(x, r)} \subset B[x, r]$.*

Proof. Since $B[x, r]$ is a closed set containing $B(x, r)$, the result follows. □

There are situations when $\overline{B(x, r)} \neq B[x, r]$. The following examples demonstrate the same.

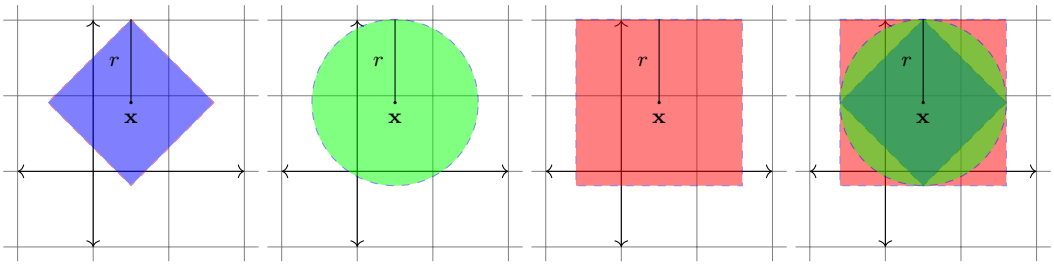
25.16. Example. In $\mathbb{R}$ with the discrete metric, $\overline{B(0, 1)} = \{0\} \neq \mathbb{R} = B[0, 1]$. In fact, in any discrete metric space X and $x \in X$, we have the following.

(1) $B(x, r) = \{x\} = B[x, r] \subset X$ for $0 < r < 1$.
(2) $B(x, 1) = \{x\} \subset B[x, 1] = X$.
(3) $B(x, r) = B[x, r] = X$ for $1 < r$.

25.17. Example. Consider the usual metric d on $\mathbb{R}$ given by $d(x, y) = |x - y|$, $(x, y \in \mathbb{R})$. Then for $x \in R$,

$$\begin{aligned} B(x, r) &= \{y \in \mathbb{R} : |x - y| < r\} \\ &= \{y \in \mathbb{R} : x - y < r, y - x < r\} \\ &= \{y \in \mathbb{R} : x - r < y < x + r\} \\ &= (x - r, x + r). \end{aligned}$$

Replacing $<$ by $\leq$ in above, we see that $B[x,r] = [x-r, x+r]$.



(a) d_1 Ball (b) d_2 Ball (c) d_∞ Ball (d) Three Balls

FIGURE 21. Open Balls in Various Metrics on $\mathbb{R}^2$

25.18. Example. Consider sum metric d_1 on $\mathbb{R}^2$ given by

$$d_1(\mathbf{x},\mathbf{y}) = |\mathbf{x}_1 - \mathbf{y}_1| + |\mathbf{x}_2 - \mathbf{y}_2|, \quad (\mathbf{x},\mathbf{y} \in \mathbb{R}^2).$$

$$\begin{aligned} B(\mathbf{x},r) &= \{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{x}_1 - \mathbf{y}_1| + |\mathbf{x}_2 - \mathbf{y}_2| < r\} \\ &= \{\mathbf{y} \in \mathbb{R}^2 : \pm(\mathbf{x}_1 - \mathbf{y}_1) \pm (\mathbf{x}_2 - \mathbf{y}_2) < r\} \\ &= \{\mathbf{y} \in \mathbb{R}^2 : \mathbf{x}_1 - \mathbf{y}_1 + \mathbf{x}_2 - \mathbf{y}_2 < r\} \\ &\quad \cap \{\mathbf{y} \in \mathbb{R}^2 : \mathbf{x}_1 - \mathbf{y}_1 - \mathbf{x}_2 + \mathbf{y}_2 < r\} \\ &\quad \cap \{\mathbf{y} \in \mathbb{R}^2 : -\mathbf{x}_1 + \mathbf{y}_1 + \mathbf{x}_2 - \mathbf{y}_2 < r\} \\ &\quad \cap \{\mathbf{y} \in \mathbb{R}^2 : -\mathbf{x}_1 + \mathbf{y}_1 - \mathbf{x}_2 + \mathbf{y}_2 < r\}, \end{aligned}$$

which is the region inside the diamond centred at $(\mathbf{x}_1, \mathbf{x}_2)$ whose vertices are r distance away along lines parallel to axes shown in the Figure 21(a).

25.19. Example. Consider the usual metric d_2 on $\mathbb{R}^2$ given by

$$d_2(\mathbf{x},\mathbf{y}) = |\mathbf{x} - \mathbf{y}| = \sqrt{|\mathbf{x}_1 - \mathbf{y}_1|^2 + |\mathbf{x}_2 - \mathbf{y}_2|^2}, \quad (\mathbf{x},\mathbf{y} \in \mathbb{R}^2).$$

$$\begin{aligned} B(\mathbf{x},r) &= \{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{x} - \mathbf{y}| < r\} \\ &= \{\mathbf{y} \in \mathbb{R}^2 : \sqrt{(\mathbf{x}_1 - \mathbf{y}_1)^2 + (\mathbf{x}_2 - \mathbf{y}_2)^2} < r\} \\ &= \{\mathbf{y} \in \mathbb{R}^2 : (\mathbf{x}_1 - \mathbf{y}_1)^2 + (\mathbf{x}_2 - \mathbf{y}_2)^2 < r^2\}, \end{aligned}$$

which is nothing but the region included inside the circle centred at $(\mathbf{x}_1, \mathbf{x}_2)$ and radius r. The ball is shown in the Figure 21(b).

25.20. Example. Consider sup metric d_∞ on $\mathbb{R}^2$ given by

$$d_\infty(\mathbf{x},\mathbf{y}) = \max\{|\mathbf{x}_1 - \mathbf{y}_1|, |\mathbf{x}_2 - \mathbf{y}_2|\}, \quad (\mathbf{x},\mathbf{y} \in \mathbb{R}^2).$$

$$\begin{aligned} B(\mathbf{x},r) &= \{\mathbf{y} \in \mathbb{R}^2 : \max\{|\mathbf{x}_1 - \mathbf{y}_1|, |\mathbf{x}_2 - \mathbf{y}_2|\} < r\} \\ &= \{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{x}_1 - \mathbf{y}_1| < r\}, \{\mathbf{y} \in \mathbb{R}^2 : |\mathbf{x}_2 - \mathbf{y}_2| < r\} \\ &= \{\mathbf{y} \in \mathbb{R}^2 : \mathbf{y}_1 \in (\mathbf{x}_1 - r, \mathbf{x}_1 + r)\}, \{\mathbf{y} \in \mathbb{R}^2 : \mathbf{y}_2 \in (\mathbf{x}_2 - r, \mathbf{x}_2 + r)\} \\ &= (\mathbf{x}_1 - r, \mathbf{x}_1 + r) \times (\mathbf{x}_2 - r, \mathbf{x}_2 + r), \end{aligned}$$

which is nothing but the region included inside the square centred at $(\mathbf{x}_1, \mathbf{x}_2)$ and having sides of length $2r$ parallel to axes. The ball is shown in the Figure 21(c).

25.21. Example. Consider $\mathbb{R}^2$ with fixed point $a = (0, 0)$. Then the open ball $B(\mathbf{x}, \frac{5}{2})$ about $\mathbf{x} = (\frac{3}{2\sqrt{2}}, \frac{3}{2\sqrt{2}})$ is as shown in the Figure 22. As it can be seen here pictorially the centre of the ball is outside the ball.

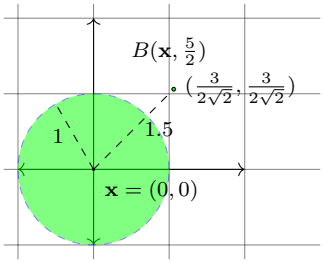


FIGURE 22. Telephone Metric Ball

25.22. Proposition. *Let (X, d) be a metric space, $a \in X$ and $r > 0$. Then for any $x \in B(x, r)$, there exists $t > 0$ such that $B(x, t) \subset B(a, r)$.*

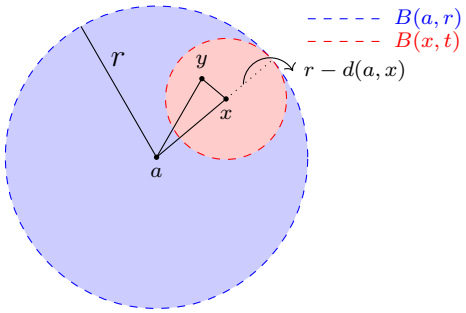


FIGURE 23. Ball Within a Ball

PROOF. Let $t = r - d(a, x)$ and $y \in B(x, t)$. Then $d(a, y) \leq d(a, x) + d(x, y) < d(a, x) + t = d(a, x) + r - d(a, x) = r$. Hence $y \in B(a, r)$. Thus $B(x, t) \subset B(a, r)$. □

25.23. Proposition. *Let (X, d) be a metric space, $a \in X$ and $r > t > 0$. Then $B(a, t) \subset B(a, r)$.*

PROOF. $x \in B(a, t) \Rightarrow d(a, x) < t < r$. Hence $x \in B(a, r)$. This proves the proposition. □

Of course, it is very important for both the open spheres to share the common centre. In the Example 25.21, one can see that pictorially the centre of a ball could be outside the ball. One more strange phenomenon is that a ball with smaller radius could properly contain a ball with larger radius. This is seen in the following example. The trick likes in changing the centre.

25.24. Example. Let $X = \{-1, 0, 1\}$ with the natural distance. Then $B(1, 2) = \{0, 1\} \subsetneq \{-1, 0, 1\} = B(0, \frac{3}{2})$.

Our next step is to prove that a topology is associated with every metric space; that is, every metric space is a topological space.

25.25. Proposition. *Let (X, d) be a metric space. Then*

$$\mathscr{B}_d = \{B_d(a, r) : a \in X, r > 0\}$$

is an open base for a topology on X.

Proof. For $x \in X$, we get $x \in B(x, 1)$. So, $X = \cup\mathscr{B}_d$. Let $r, t > 0$, $a, b \in X$, and $x \in B(a, r) \cap B(b, t)$. Then by the Proposition 25.22, there exist $s_1, s_2 > 0$ such that $B(x, s_1) \subset B(a, r)$ and $B(x, s_2) \subset B(a, t)$. Taking $s = \min\{s_1, s_2\}$ and applying the Proposition 25.23, we see that $B(x, s) \subset B(a, r) \cap B(b, t)$. This completes the proof. □

25.26. Definition. Let (X, d) be a metric space the topology generated by the base $\mathscr{B}_d$ in the Proposition 25.25 is called *the topology induced by the metric* d and is denoted by $\mathscr{T}_d$. In general, a topology induced by a metric is called a *matric topology* or a *metrizable topology.*

The following is easy to prove.

25.27. Proposition. *Let (X, d) be a metric space and $x \in X$. Then the following are local bases at x.*

(1) $\{B(x, r) : r > 0\}$.
(2) $\{B(x, r) : r \in \mathbb{Q}, r > 0\}$.
(3) $\{B(x, \frac{1}{n}) : n \in \mathbb{N}\}$.

25.28. Remark. For an element x of a metric space, $\{B(x, \frac{1}{n}) : n \in \mathbb{N}\}$ is a countable local base at x, and hence, every metric space is first countable.

25.29. Remark. Let us note that a metric space is a particular case of a topological space. Thus we have the following for a metric space (X, d).

(1) In view of the Proposition 25.25, $G \subset X$ is open in X if and only if for every $x \in G$, there exists $r > 0$ such that $B(x, r) \subset G$ if and only if G is a union of open balls if and only if for every $x \in G$, there exist $a \in X$ and $t > 0$ such that $x \in B(a, t) \subset G$.
(2) We denote the topology generated by the metric d by $\mathscr{T}_d$.

Figure 24. Intersecting Balls

(3) If we begin with a topology $\mathscr{T}$ on a the set X, it is obvious to ask a question, "Is there any metric ρ defined on X such that $\mathscr{T} = \mathscr{T}_\rho$"? If there exists such a metric, then we say that the topology $\mathscr{T}$ is *metrizable*.
(4) When we refer to a set X as a metric space, we assume that the underlying metric is denoted by d.
(5) Whenever we consider $\mathbb{R}^n$ without reference to a particular metric, we mean everything with respect to the usual metric.

25.30. Definition. Two metrics d and ρ on a set X are said to be *equivalent* if they induce the same topology on X.

25.31. Proposition. *Let (X, d) be a metric space. Define $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$, $(x, y \in X)$ as in (25.10.2). Then d and ρ are equivalent.*

Proof. Let $x \in X$. Let $\varepsilon > 0$. Consider $B_\rho(x, \varepsilon)$. If $\varepsilon > 1$, then $B_d(x, \varepsilon) \subset X = B_\rho(x, \varepsilon)$. If $\varepsilon \leq 1$, let $r = \frac{\varepsilon}{1+\varepsilon}$. Then

$$\begin{aligned}
y \in B_d(x, r) &\Rightarrow d(x, y) < r = \frac{\varepsilon}{1+\varepsilon} \\
&\Rightarrow 1 - \frac{\varepsilon}{1+\varepsilon} < 1 - d(x, y) \\
&\Rightarrow \frac{1}{1+\varepsilon} < 1 - d(x, y) < 1 + d(x, y) \\
&\Rightarrow \frac{1}{1+d(x,y)} < (1+\varepsilon) \\
&\Rightarrow \frac{d(x,y)}{1+d(x,y)} < \frac{\varepsilon}{1+\varepsilon}(1+\varepsilon) = \varepsilon \\
&\Rightarrow \rho(x, y) < \varepsilon.
\end{aligned}$$

Thus $B_d(x, r) \subset B_\rho(x, \varepsilon)$. On the other hand, consider $B_d(x, \varepsilon)$. Let $r = \frac{\varepsilon}{1+\varepsilon}$

$$\begin{aligned}
x \in B_\rho(x, r) &\Rightarrow \frac{d(x,y)}{1+d(x,y)} < r \\
&\Rightarrow d(x, y) < r + rd(x, y) \\
&\Rightarrow d(x, y) - rd(x, y) < r \\
&\Rightarrow d(x, y) < \frac{r}{1-r} \\
&\Rightarrow d(x, y) < \frac{\frac{\varepsilon}{1+\varepsilon}}{1 - \frac{\varepsilon}{(1+\varepsilon)}} = \varepsilon.
\end{aligned}$$

Now let $G \subset X$ be $\mathscr{T}_d$-open and $x \in G$. So, there is $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subset G$. Taking $r = \frac{\varepsilon}{1+\varepsilon}$, we get $B_\rho(x, r) \subset B_d(x, \varepsilon) \subset G$. Thus G is open in $\mathscr{T}_\rho$.

Conversely, suppose $G \subset X$ is $\mathscr{T}_\rho$-open and $x \in G$. So, there is $\varepsilon > 0$ such that $B_\rho(x, \varepsilon) \subset G$. Taking $r = \frac{\varepsilon}{1+\varepsilon}$, we get $B_d(x, r) \subset B_\rho(x, \varepsilon) \subset G$. Thus G is open in $\mathscr{T}_d$. Thus $\mathscr{T}_d = \mathscr{T}_\rho$. □

25.32. Proposition. *Let d, ρ be two metrics on a set X. Suppose there are $M, N > 0$ such that*

$$M\rho(x,y) \le d(x,y) \le N\rho(x,y)$$

for all $x, y \in X$. Then d and ρ are equivalent.

PROOF. For $x, y \in X$,

$$\begin{aligned} y \in B_d(x, M\varepsilon) &\Rightarrow d(x,y) < M\varepsilon \\ &\Rightarrow M\rho(x,y) < d(x,y) < M\varepsilon \\ &\Rightarrow \rho(x,y) < \varepsilon \\ &\Rightarrow y \in B_\rho(x,\varepsilon). \end{aligned}$$

Also,

$$\begin{aligned} y \in B_\rho(x, \frac{\varepsilon}{N}) &\Rightarrow \rho(x,y) < \frac{\varepsilon}{N} \\ &\Rightarrow d(x,y) < N\rho(x,y) < N\frac{\varepsilon}{N} \\ &\Rightarrow d(x,y) < \varepsilon \\ &\Rightarrow y \in B_d(x,\varepsilon). \end{aligned}$$

Consequently, $B_d(x, M\varepsilon) \subset B_\rho(x,\varepsilon)$ and $B_\rho(x, \frac{\varepsilon}{N}) \subset B_d(x,\varepsilon)$. Thus d and ρ generate the same topology on X and hence are equivalent. □

In general, the converse of Proposition 25.32 is not true.

25.33. Example. On $\mathbb{R}$ consider the usual metrics $d(x,y) = |x-y|$, and $\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$, $(x, y \in \mathbb{R})$. Then by the Proposition 25.31, d and ρ are equivalent. However, there is no $N > 0$ such that $d(x,y) \le N\rho(x,y)$.

We shall characterize the equivalent metrics on a set using convergent sequences in the next section.

Problems

170. Let d, ρ be two metrics on a set X. Define for $x, y \in X$,

$$\psi_1(x,y) = d(x,y) + \rho(x,y); \tag{25.33.1}$$

$$\psi_2(x,y) = \max\{d(x,y), \rho(x,y).\} \tag{25.33.2}$$

Show that ψ_1, ψ_2 are metrics on X.

171. Let (X,d) be a metric space and Y be a set. Let $f\colon Y \to X$ be a bijective function. Define for $y_1, y_2 \in Y$, $\rho(y_1,y_2) = d(f(y_1), f(y_2))$. Show that ρ is a metric on Y.

172. Combining the proof of the Examples 25.9 and 25.10, prove that on a metric space (X,d), for $r > 0$, $\sigma_r(x,y) = \frac{d(x,y)}{1+rd(x,y)}$ $(x, y \in X)$, defines a metric on X.

173. Show that d_T defined in the Example 25.11 is a metric on X.

174. Let $X = \mathbb{R}^2$. For $\mathbf{x}, \mathbf{y} \in X$, define

$$d(\mathbf{x},\mathbf{y}) = \begin{cases} 0, & \text{if } \mathbf{x} = \mathbf{y}; \\ |\mathbf{x}| + |\mathbf{y}|, & \text{otherwise.} \end{cases}$$

Find $B((0,0),1)$, $B((1,1),\sqrt{2})$, $B((1,1),2$. Sketch them.

175. Consider $X = \{(0,0)\} \cup \{(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Show that $B((0,0), 0.4) = B((0,0), 0.5)$. Sketch $B((1,0), 0.5)$.

176. Let (X, d) be a metric space and $A \subset X$ be dense in A. Let $\mathscr{B}_x$ be a local base at each $x \in A$. Define

$$\mathscr{B} = \cup\{\mathscr{B}_x : x \in A\}.$$

Show that $\mathscr{B}$ is an open base for $(X, \mathscr{T}_d)$. Also, show that we cannot drop the condition that X is a metric space. (see the Example 11.5(2).)

177. Let (X, d) be a metric space and $A \subset (0, \infty)$. Show that $\{b(x, r) : r \in A\}$ is a local base at each $x \in X$ if and only if $0 \in \overline{A}$.

26. Sequences

In this section, we define and obtain properties of sequences. We define and obtain the properties of sequences. Wherever possible, we obtain the results in a general topological space. But certain concepts and results do not make sense in a general topological space so they are restricted to the metric spaces.

26.1. Definition. A *sequence* in a set X is a function $n \in \mathbb{N} \mapsto x_n \in X$ and is denoted by (x_n) or $\{x_n\}$. The set $\{x_n : n \in \mathbb{N}\}$ is said to be the *range* of the sequence $\{x_n\}$.

We shall need the following terminology in the sections to follow.

26.2. Definition. Let $\{x_n\}$ be a sequence in a set X. A *subsequence* of $\{x_n\}$ is a sequence $\{x_{n_k}\}$, where $n_1 < n_2 < n_3 < \cdots$. For $N \in \mathbb{N}$, we define $T_N(\{x_n\})$ to be the set $\{x_N, x_{N+1}, \dots\}$, called a *tail of the sequence* $\{x_n\}$. The sequence $\{x_n\}$ is said to be *constant* if $x_n = x_m$ for all $n, m \in \mathbb{N}$. $\{x_n\}$ is said to be *eventually constant* if one of its tails is a constant sequence. Let $A \subset X$. We say that $\{x_n\}$ is *frequently in* A if A contains a subsequence of $\{x_n\}$, that is, for every $n \in \mathbb{N}$, there is $m \geq n$ such that $x_m \in A$. We say that $\{x_n\}$ is *eventually in* A if A contains a tail of $\{x_n\}$.

If range of a sequence is a finite set, then we call that sequence a *sequence with finite range* otherwise it is called a *sequence with infinite range*.

Now we turn to the concepts of a Cauchy sequence and a convergent sequence. The concept of a Cauchy sequence is not available in a general topological space. It requires a metric or a uniformity structure. We limit ourselves with the metric structure. We include the uniform spaces in the latter sections of the notes.

26.3. Definition. Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is

(1) *convergent* in X if there exists $x \in X$ such that for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ whenever $n \geq N$. In this case, we write $x_n \to x$ in X or $\lim_{n\to\infty} x_n = x$.

(2) *Cauchy* if for every $\varepsilon > 0$, there is a $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$.

26.4. Proposition. *Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X and $x \in X$.*

(1) $\{x_n\}$ is Cauchy if and only if $d(x_n, x_m) \to 0$ and $n, m \to \infty$.
(2) $x_n \to x$ is convergent if and only if $d(x_n, x) \to 0$ and $n \to \infty$.

Proof. ☞(1) $\{x_n\}$ is Cauchy if and only if given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for every $n, m > N$ if and only if there is $N \in \mathbb{N}$ such that $|d(x_n, x_m) - 0| < \varepsilon$ for every $n, m > N$ if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

☞(2) $x_n \to x$ if and only if given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for every $n > N$ if and only if there is $N \in \mathbb{N}$ such that $|d(x_n, x) - 0| < \varepsilon$ for every $n > N$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. □

Figure 25. Cauchy Sequence

Look at the adjacent Figure 25. Think of an ant roaming on the largest square. As soon as it enters from red to the green zone, say the four sides of the green square, are blocked not allowing the ant to come to the red zone. Now as soon as it enters the blue square, again the entry to the green square is restricted. If we go on doing this, and thereby restricting the area the ant can move. Continuing in this way, we go on restricting the movement of the ant in the smaller and smaller region. You will say that ultimately the ant will be restricted to be steadily on the point drawn at the centre. If we mark the position of the ant at every second, then we will infer that the dots are accumulating near the centre. In the long run, the dots come nearer to each other. Such positions of the dots, is what a Cauchy sequence is. In long run, the terms of the sequence are near to each other. Eventually going towards a point, in the case of our ant, that point is the centre. So, we say that the ant is *converging to the centre*. Now suppose that in our experiment, we have started with a piece of cardboard in which there is a hole at the centre. So the ant eventually goes towards some point which does not exist on the cardboard. However, any of her two states in long run are very nearer to each other, though they are aimless! This represents a Cauchy sequence which wants to converge, *but its aim is not in the space*. The following proposition and example show exactly the same.

26.5. Proposition. *Every convergent sequence is Cauchy.*

PROOF. Let $\{x_n\}$ be a sequence in (X,d) converging to x. Let $\varepsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \frac{\varepsilon}{2}$ for every $n \geq \mathbb{N}$. But then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

26.6. Theorem. *A Cauchy sequence is convergent if and only if it has a subsequence that converges.*

PROOF. Let $\{x_n\}$ be a Cauchy sequence in a metric space. Suppose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $x \in X$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ whenever $n, m \geq N$. Choose $K \in \mathbb{N}$ such that $d(x_{n_k}, x) < \frac{\varepsilon}{2}$ whenever $k \geq K$. Let $M = \max\{n_K, N\}$. Then for $n \geq M$, $d(x_n, x) \leq d(x_n, x_{n_K}) + d(x_{n_K}, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Thus $x_n \to x$. On the other hand, a convergent sequence is its own convergent subsequence. This completes the proof. $\square$

FIGURE 26. Nonconvergent Cauchy Sequence

26.7. Example. A Cauchy sequence need not converge. Indeed let $X = \mathbb{R} \setminus \{0\}$ with the usual metric. Then the sequence $\{\frac{1}{n}\}$ is Cauchy but not convergent.

Indeed let $\varepsilon > 0$ be given. Choose $N > \varepsilon$. Then for $n, m > N$, assuming that $n > m$, $|\frac{1}{m} - \frac{1}{n}| < \frac{1}{m} < \varepsilon$. Thus $\{\frac{1}{n}\}$ is Cauchy. Suppose $\frac{1}{n} \to x$. Since $0 \notin \mathbb{R} \setminus \{0\}$, $x \neq 0$. If $x < 0$, then taking $\varepsilon = |x|$, we see that $|x - \frac{1}{n}| = |x| + \frac{1}{n} \geq \varepsilon$ for every n. For $x > 0$, choose $k \in \mathbb{N}$ such that $\frac{1}{x} < k$ as in the Figure 26. Let $\varepsilon = \frac{1}{k} - \frac{1}{k+1}$. Then for every $n \geq k+1$, $|x - \frac{1}{n}| > \varepsilon$. Thus $\frac{1}{n} \not\to x$. This shows that $\{\frac{1}{n}\}$ is Cauchy but not convergent in $\mathbb{R} \setminus \{0\}$.

We can generalize the concept of convergent sequences in arbitrary topological space using the following characterization. However, the concept of Cauchy sequences can be generalized only up to the uniform spaces which we postpone to the Chapter 10.

26.8. Lemma. *Let (X,d) be a metric space, $\{x_n\}$ be a sequence in X and $x \in X$. The following are equivalent.*

(1) $x_n \to x$.
(2) Every $U \in \mathscr{U}_x$ contains a tail of $\{x_n\}$.

PROOF. ☞(1) ⇒ (2) Assume that $x_n \to x$. Let $U \in \mathscr{U}_x$. So, $x \in U^\circ \subset U$; consequently there is $\varepsilon > 0$ such that $B(x,\varepsilon) \subset U$. Since $x_n \to x$, there is $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for every $n \geq N$. That is, $T_N\{x_n\} \subset B(x,\varepsilon) \subset U$.

☞(2) ⇒ (1) Assume (2). Let $\varepsilon > 0$. Then $B(x,\varepsilon)$, being a neighbourhood of x, contains a tail $T_N(\{x_n\})$. That is, $d(x_n, x) < \varepsilon$ for every $n \geq N$. Thus $x_n \to x$. $\square$

Using this we define the following.

26.9. Definition. Let X be a topological space and $\{x_n\}$ be a sequence in X and $x \in X$. We say that $x_n \to x$ if every $U \in \mathscr{U}_x$ contains a tail of $\{x_n\}$.

26.10. Proposition. *Subsequence of a convergent sequence converges to the same limit(s) of the sequence.*

Proof. Let X be a topological space and $x_n \to x$ in X. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. Let $U \in \mathscr{U}_x$ contain the tail $T_m(\{x_n\})$ of $\{x_n\}$. Choose $k \in \mathbb{N}$ such that $n_k \geq m$. Then $i \geq k \Rightarrow x_{n_k} \in U$. Thus $x_{n_k} \to x$. □

26.11. Proposition. *An eventually constant sequence is convergent but not conversely.*

Proof. Let $\{x_n\}$ be an eventually constant sequence. So one of its tails, say $T_N(\{x_n\})$ is a constant sequence $x, x, x, \ldots$. Let $U \in \mathscr{U}_x$. Then $T_N(\{x_n\}) \subset U$. Hence $x_n \to x$.

Note that in $\mathbb{R}$, $\frac{1}{n} \to 0, \frac{n}{n+2} \to 1$, none of which is eventually constant. In fact, the all their terms are distinct. □

Sequences behave well and reveal much about a metrizable topology. However it need not be the case in arbitrary topological space.

26.12. Proposition. *A sequence in a metric space can converge to at most one point.*

Proof. Let (X, d) be a metric space and $\{x_n\}$ be a convergent sequence. Suppose $x_n \to x$ as well as $x_n \to y$, where $x \neq y$. But then $r = d(x, y) > 0$. So that $B(x, \frac{r}{2})$ contains a tail $T_N(\{x_n\})$ as well as $B(y, \frac{r}{2})$ contains a tail $T_M(\{x_n\})$. Note that x_{N+M} is a part of both the tails. Thus

$$r = d(x, y) \leq d(x_{N+M}, x) + d(x_{N+M}, y) < \frac{r}{2} + \frac{r}{2} = r,$$

a contradiction. Thus $x = y$. □

26.13. Example. A sequence in a topological space, need not have a unique limit. The sequence $\{n\}$ in $\mathbb{R}$ with the cofinite topology converges to every point of $\mathbb{R}$. Indeed for any $x \in \mathbb{R}$, and $U \in \mathscr{U}_x$, there will be possibly only finitely many $a_1, a_2, \ldots a_k$ outside U. Taking an integer $N > \max\{a_1, a_2, \ldots, a_k\}$, we see that $N, N+1, \ldots \in U$. Thus $x_n \to x$.

The following concept is not available in a general topological space.

26.14. Definition. Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is said to be bounded if there is $M > 0$ such that $d(x_n, x_m) \leq M$ for every $n, m \in \mathbb{N}$.

26.15. Theorem. *A Cauchy sequence is bounded.*

PROOF. Let $\{x_n\}$ be a Cauchy sequence in a metric space (X, d). Then there is $N \in \mathbb{N}$ such that $d(x_n, x_m) < 1$ for every $n, m \geq N$. Let $M = \max\{d(x_n, x_m) + 1 : n, m \leq N\}$. Clearly $d(x_n, x_m) \leq M$ for every $n, m \in \mathbb{N}$. Thus $\{x_n\}$ is bounded. □

26.16. Corollary. *A convergent sequence is bounded.*

26.17. Example. A bounded sequence need not be Cauchy. Indeed $\{(-1)^n\}$ is a bounded sequence in $\mathbb{R}$ which is not Cauchy.

26.18. Proposition. *Let $\{x_n\}, \{y_n\}$ be sequences in $\mathbb{R}$.*

(1) $x_n \to x \Rightarrow -x_n \to -x$.
(2) $x_n \to x \Rightarrow |x_n| \to |x|$ *but the converse does not hold.*
(3) $x_n \to x \Rightarrow \frac{1}{x_n} \to \frac{1}{x}$ *provided* $x_n \neq 0$ *for every* n *and* $x \neq 0$.
(4) $x_n \to x, y_n \to y \Rightarrow x_n + y_n \to x + y$.
(5) $x_n \to x, y_n \to y \Rightarrow x_n - y_n \to x - y$.
(6) $x_n \to x, y_n \to y \Rightarrow x_n y_n \to xy$.
(7) $x_n \to x, y_n \to y \Rightarrow \frac{x_n}{y_n} \to \frac{x}{y}$ *provided* $y_n \neq 0$ *for every* n *and* $y \neq 0$.

PROOF. ☞(1) This follows from $|x_n - x| = |-x_n + x|$.

☞(2) This follows from $||x_n| - |x|| \leq |x_n - x| < |\frac{x}{2}|$ for large enough n.

☞(3) By (2), there is $N \in \mathbb{N}$ such that $|x_n| > \frac{|x|}{2}$ for $n \geq N$. Thus $\frac{1}{|x_n|} < \frac{2}{|x|}$. Now given $\varepsilon > 0$, let $\delta = \frac{|x|^2 \varepsilon}{2}$. Choose $K \in \mathbb{N}$ such that $|x_n - x| < \delta$ for every $n \geq K$.

$$\left|\frac{1}{x_n} - \frac{1}{x}\right| = \left|\frac{x - x_n}{x_n x}\right| < \left|\frac{2(x_n - x)}{|x|^2}\right| < \frac{2\delta}{|x|^2} = \varepsilon,$$

for every $n \geq K$.

☞(4) Let $\varepsilon > 0$. Then choose $N, M \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{2}$ whenever $n \geq N$ and $|y_n - y| < \frac{\varepsilon}{2}$ whenever $n \geq M$. Let $K = \max\{M, N\}$, then for every $n \geq K$,

$$|x_n + y_n - x - y| \leq |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}.$$

☞(5) This follows by the combination of (1) and (4).

☞(6) Since $y_n \to y$, there is $M \geq 0$ such that $|y_n| < M$. Given $\varepsilon > 0$, choose $N, K \in \mathbb{N}$ such that $|x_n - x| < \frac{\varepsilon}{2M}$ for $n \geq N$ and $|y_n - y| < \frac{\varepsilon}{2(|x|+1)}$ whenever $n \geq K$. Then for $n \geq \max\{N, K\}$,

$$|x_n y_n - xy| \leq |x_n y_n - x y_n + x y_n - xy| < |y_n||x_n - x| + |x||y_n - y| < \varepsilon.$$

☞(7) follows by combining (3) and (6). □

26.19. Example. Let (X, d) be a metric space and $a \in X$ be fixed. Let d_T denote the telephone metric on X defined as in Example 25.11, that is

$$d_T(x,y) = \begin{cases} 0, & \text{if } x = y; \\ d(x,a) + d(a,y), & \text{if } x \neq y, \end{cases}$$

Then a sequence $\{x_n\}$ converges to $x \neq a$ with respect to d_T if and only if $\{x_n\}$ is eventually constant to a. Also, a sequence $x_n \to a$ with respect to d_T if and only if $x_n \to a$ with respect to d.

26.20. Example. A sequence in an indiscrete space coverges to all points. Indeed, let X be an indiscrete space and $\{x_n\}$ be a sequence in X. Let $x \in X$. Then $\mathscr{U}_x = \{X\}$ and X contains the full tail of $\{x_n\}$. Thus $x_n \to x$.

26.21. Example. A sequence in a discrete space converges if and only if it is eventually constant. Clearly, in view of the Proposition 26.11, an eventually constant sequence in a discrete space converges. Conversely, suppose X is a discrete space and $\{x_n\}$ is a sequence in X converging to $x \in X$. So, $\{x\} \in \mathscr{U}_x$ contains a tail of $\{x_n\}$, forcing it to be eventually constant.

26.22. Example. Let X be a cofinite space and $\{x_n\}$ be a sequence in X. Suppose no subsequence of $\{x_n\}$ is constant. Then $x_n \to x$ for every $x \in X$. Indeed, let $U \in \mathscr{U}_x$. Then $X \setminus U$ is finite, say $X \setminus U = \{a_1, a_2, \ldots, a_m\}$. Since none of the a_i's can occur in $\{x_n\}$ infinitely many times, there is $N \in \mathbb{N}$ such that $n \geq N \Rightarrow x_n \neq a_k$ for $1 \leq k \leq m$. Thus $T_N(\{x_N\}) \subset U$. Hence, $x_n \to x$.

If $\{x_n\}$ contains exactly one constant subsequence $\{x, x, x, \ldots\}$, then $x_n \to x$ and $x_n \not\to y$ for $y \neq x$. Indeed let $U \in \mathscr{U}_x$. Let $X \setminus U = \{a_1, a_2, \ldots, a_m\}$. Then there is $N \in \mathbb{N}$ such that $n \geq N \Rightarrow x_n \neq a_k$ for $1 \leq k \leq m$. Thus $T_N(\{x_N\}) \subset U$. Hence, $x_n \to x$. Also, for $y \neq x$, $U - X \setminus \{x\}$ does not contain any tail of $\{x_n\}$ as x repeats infinitely many times.

Now it follows that if $\{x_n\}$ contains more than one constant subsequence, then $\{x_n\}$ does not converge to any point of X.

26.23. Example. The convergent sequences in a cocountable space are precisely eventually constant sequences. Indeed, let $\{x_n\}$ be a sequence in a cocountable space. As we have seen if $\{x_n\}$ is an eventually constant sequence, then it converges. Suppose $x_n \to x$. Let $A = \{x_n : n \in \mathbb{N}\}$ be the range space. Let $U = (X \setminus A) \cup \{x\}$. Then $U \in \mathscr{U}_x$. So, $T_N(\{x_n\}) \subset U$ for some N. But $x_n \in U$ for every $n \geq N$ implies $x_n = x$ for every $n \geq N$. Thus $\{x_n\}$ is eventually constant. The converse follows from the Proposition 26.11.

26.24. Proposition. *Let X have two topologies $\mathscr{T}_1$, $\mathscr{T}_2$ with $\mathscr{T}_1 \leq \mathscr{T}_2$ and $\{x_n\}$ be a sequence in X. If $x_n \to x$ in $(X, \mathscr{T}_2)$, then $x_n \to x$ in $(X, \mathscr{T}_1)$.*

Proof. Indeed, let U be a $\mathscr{T}_1$-neighbourhood of x. So, U is a $\mathscr{T}_2$-neighbourhood of x. Hence U contains a tail of $\{x_n\}$. This completes the proof. □

Note that on a set X $\mathscr{I} \leq \mathscr{D}$ but a eventually nonconstant sequence in X converges in $(X, \mathscr{I})$ but does not converge in $\mathscr{D}$.

26.25. Proposition. *Let X be a set and $\mathscr{T}$ be a topology on X such that $\mathscr{T}_c \leq \mathscr{T}$. Then the convergent sequences in $(X, \mathscr{T})$ are precisely the eventually constant sequences.*

Proof. Let $x_n \to x$ in $(X, \mathscr{T})$. Then by the Proposition 26.24, $x_n \to x$ in $(X, \mathscr{T}_c)$. But then $\{x_n\}$ must be eventually constant. In view of the Proposition 26.11, the proof is complete. □

26.26. Corollary. *Let α be a cardinal number such that $\aleph_0 < \alpha$. Then the convergent sequences in $\mathscr{T}_{<\alpha}$ and $\mathscr{T}_\alpha$ are precisely the eventually constant sequences.*

26.27. Example. Consider a set X a point $a \in X$ and the VIP topology $\mathscr{T} = \{U \subset X : U = \emptyset \text{ or } a \in U\}$. Let $b \in X$. Let $\{x_n\}$ be a sequence in X. Then $x_n \to b$ if and only if there is $N \in \mathbb{N}$ such that $\{x_n : n \geq N\} \subset \{a, b\}$.

26.28. Example. Consider a set X a point $a \in X$ and the excluded point topology $\mathscr{T} = \{U \subset X : U = X \text{ or } a \notin U\}$. Let $b \in X \setminus \{a\}$. Let $\{x_n\}$ be a sequence in X. Then

(1) $x_n \to b$ if and only if $\{x_n\}$ eventually constants to b; and
(2) $x_n \to a$.

26.29. Example. Let X be a set and $A \subset X$. A sequence $\{x_n\}$ in X converges to all points of $X \setminus A$. However it converges to one, and hence all, points of A if and only if it is eventually in A. Note that this gives the convergence in an indiscrete space by taking $A = \emptyset$.

26.30. Example. Let X be a nonempty set and $\{A_i : 1 \leq i \leq n\}$ be a chain of subsets of X. Consider the topology $\mathscr{T} = \{\emptyset, X, A_1, A_2, \ldots, A_n\}$ as in Example 7.18. To avoid the triviality, we assume that $\emptyset \neq A_1 \subsetneqq A_2 \subsetneqq \cdots \subsetneqq A_n \subsetneqq X$. Every sequence converges to all the points of $X \setminus A_n$. Also, a sequence $\{x_n\}$ converges to one, and hence all, points of A_k if and only if $\{x_n\}$ is eventually in some A_j for $j \leq k$.

26.31. Example. For a set X and $\emptyset \subsetneqq A \subsetneqq X$, consider $\mathscr{T} = \{\emptyset, X, A, X \setminus A\}$ is a topology on X. A sequence $\{x_n\}$ converges to a point, and hence all points, of A if and only if it is eventually in A and like wise to a point, and hence all points, of $X \setminus A$ if and only if it is eventually in $X \setminus A$.

26.32. Example. For a set X and $A \subset X$, consider the topology $\mathscr{T} = \{G \subset X : G = \emptyset \text{ or } A \subset G\}$ on X. A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if it is eventually in $A \cup \{x\}$. In particular, a sequence eventually in A converges to all points.

26.33. Example. For a set X, $A \subset X$, consider the topology $\mathscr{T} = \{G \subset X : G = X \text{ or } G \subset A\}$ on X. Let $x \in X \setminus A$. Then every sequence converges to x. Let $x \in X$. Then a the only sequence converging to x is the sequence that eventually constants to x.

26.34. Proposition. *A sequence $\{x_n\}$ converges to $x \in \mathbb{R}_\ell$ if and only if every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a monotonically decreasing subsequence $\{x_{n_{k_j}}\}$ such that $\inf\{x_{n_{k_j}}\} = x$.*

Proof. Suppose $x_n \to x$ in $\mathbb{R}_\ell$. If $\{x_n\}$ is eventually constant x, then we are done. So, we assume that it is not the case. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. By Proposition 26.10, $x_{n_k} \to x$. Hence $[x, x+1)$ contains a tail, say $T_{k_1}(\{x_{n_k}\})$. Without loss of generality, we assume that $x_{n_{k_1}} \neq x$. But then by the convergence of $T_{k_1}(\{x_{n_k}\})$ to x, there is $k_2 > k_1$ such that $T_{k_2}(\{x_{n_k}\})$ is contained in $[x, s_1)$, where $s_1 = \min\{x + \frac{1}{2}, x_{n_{k_1}}\}$. Thus $x_{n_{k_2}} < x_{n_{k_1}}$. We choose $k_3 > k_2$ such that such that $T_{k_3}(\{x_{n_k}\})$ is contained in $[x, s_2)$, where $s_2 = \min\{x + \frac{1}{3}, x_{n_{k_2}}\}$. Continuing, in this way we obtain a monotonically decreasing subsequence $x_{n_1} > x_{n_2} > x_{n_3} > \dots$ of $\{x_{n_k}\}$. Also, $x \leq x_{n_{k_j}} \leq x + \frac{1}{j}$. Hence $\inf x_{n_{k_j}} = x$.

Conversely, suppose every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a monotonically decreasing subsequence $\{x_{n_{k_j}}\}$ such that $\inf\{x_{n_{k_j}}\} = x$. Suppose if possible, that $x_n \not\to x$. Hence there is $r > 0$ and a subsequence x_{n_k} of $\{x_n\}$ such that $x_{n_k} > x + r$ for each $k > 0$. But then $\inf\{x_{n_k}\} \geq x + r$. So, none of it's subsequences has x as its infimum. □

Applying the symmetric arguments as in the above proposition, we have the following. Recall that $\mathbb{R}_u$ is $\mathbb{R}$ with the upper limit topology.

26.35. Proposition. *A sequence $\{x_n\}$ converges to $x \in \mathbb{R}_u$ if and only if every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ has a monotonically increasing subsequence $\{x_{n_{k_j}}\}$ such that $\inf\{x_{n_{k_j}}\} = x$.*

The following is a consequence of the Proposition 8.25.

26.36. Proposition. *A sequence $\{x_n\}$ in $(\mathbb{Z}, \mathscr{T}_{ap})$ converges to $m \in \mathbb{Z}$ if and only if $\{x_n - m\}$ converges to 0.*

Proof. Clearly, $B_{a,b} \in \mathscr{U}_m$ if and only if $B_{a,b} - m \in \mathscr{U}_0$, where $B_{a,b} - m = \{k - m : k \in B_{a,b}\}$. □

In light of Proposition 26.36, it is sufficient to obtain the description of sequences converging to 0.

The following is apparent be because the basic neighbourhoods of 0 are of the form $m\mathbb{Z}$, $(m \in \mathbb{N})$.

26.37. Proposition. *A sequence $\{x_n\}$ in $(\mathbb{Z}, \mathscr{T}_{ap})$ converges to 0 if and only if for every $m \in \mathbb{N}$, there is $k > 0$ such that $m \mid x_n$ for every $n \geq k$.*

26.38. Example. [LM15, p.108] If $x_n = n!$, $y_n = \operatorname{lcm}(1, \dots, n)$, then $x_n \to 0$ and $y_n \to 0$ in $(\mathbb{Z}, \mathscr{T}_{ap})$.

26.39. Example. A sequence $x_n \to x$ in $\mathbb{R}_K$ if and only if it does so in $\mathbb{R}$ with the usual topology and $\{x_n\} \cap K$ is finite.

26.40. Example. Let X be a set, $Y \subset X$ and $\mathscr{T}$ be a topology on Y and $\mathscr{T}^{Y\mathscr{D}}$ be the discrete extension of $\mathscr{T}$ on X defined in Proposition 12.10. A sequence $\{x_n\}$ in X converges to $x \in Y$ if and only if Y contains a tail $T_k\{x_n\}$ of $\{x_n\}$ and $x_{n+k} \to x$ in Y. On the other hand $x_n \to x$ for some $x \in X \setminus Y$ if and only if $\{x_n\}$ eventually constants to x.

26.41. Example. Let X be a set, $Y \subset X$ and $\mathscr{T}$ be a topology on Y and $\mathscr{T}^{Y\mathscr{I}}$ be the indiscrete extension of $\mathscr{T}$ on X defined in Proposition 12.11. A sequence $\{x_n\}$ in X converges to $x \in Y$ if and only if Y contains a tail $T_k\{x_n\}$ of $\{x_n\}$ and $x_{n+k} \to x$ in Y. On the other hand $x_n \to x$ for all $x \in X \setminus Y$.

26.42. Proposition. *Let X be a totally ordered set with the right order topology. A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if there is a sequence $a_1 \le a_2 \le a_3 \le \cdots \le x$ such that $\sup\{a_k\} = x$ and each $(a_k\infty)$ contains a tail of $\{x_n\}$.*

Proof. Suppose there is a sequence $a_1 \ge a_2 \ge a_3 \ge \cdots \ge x$ such that $\sup\{a_k\} = x$ and each $(a_k\infty)$ contains a tail of $\{x_n\}$. Let $(a, \infty) \in \mathscr{U}_x$. Since $a < x$, a is not an upper bound of $\{a_n\}$. Thus for some $k \in \mathbb{N}$, $a < a_k$. Hence (a, ∞) contains a tail of $\{x_n\}$. Thus $x_n \to x$.

Conversely, suppose $x_n \to x$. If $[x, \infty)$ contains a tail of $\{x_n\}$, then we are done. So, assume that $[a, \infty)$ does not contain any tail of $\{x_n\}$. If a is the immediate predecessor of x, then $[x, \infty) = (a, \infty)$ contains a tail of $\{x_n\}$, a contradiction. Thus x cannot have the immediate predecessor. Now let $a < x$. Since (a, ∞) contains a tail of $\{x_n\}$, and since that tail will not be in $[x, \infty)$, there is $x_{n_1} \in (a, x)$. Replacing a by x_1, we get a tail of $\{x_n\}$ in (x_{n_1}, x) and $n_2 > n_2$ such that $x_{n_2} \in (x_{n_1}, x)$. Repeating the argument inductively, we get $n_1 < n_2 < n_2 < \cdots < x$ such that $x_{n_1} < x_{n_2} < x_{n_2} < \cdots < x$ such that each (x_{n_k}, x) contains a tail of $\{x_n\}$. Since $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, $x_{n_k} \to x$. This, together with the monotonicity of $\{x_{n_k}\}$, it follows that $\sup\{x_{n_k}\} = x$. Taking $a_k = x_{n_k}$, the result follows. □

The following is at once now using the similar argument.

26.43. Proposition. *Let X be a totally ordered set with the left order topology. A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if there is a sequence $a_1 \ge a_2 \ge a_3 \ge \cdots \ge x$ such that $\inf\{a_k\} = x$ and each $(-\infty, a_k)$ contains a tail of $\{x_n\}$.*

26.44. Example. Let (X, d) be a metric space, $a \in X$ and d_T be the telephone metric defined as in Example 25.11. Then a sequence converges to a point other than a if and only if it is essentially constant sequence. Also, a sequence converges to a if and only if it converges to a in (X, d).

Problems

178. Prove the following.

(1) Homeomorphic image of a Cauchy sequence need not be a Cauchy sequence.
(2) Homeomorphic image of a convergent sequence is a convergent sequence.

179. In the following $\mathbb{R}$ carries the usual metric.

(1) Show that $\frac{1}{n} \to 0$ in $\mathbb{R}$.
(2) Let $0 \leq a_n \leq \frac{1}{n}$ for every $n \in \mathbb{N}$. Show that $a_n \to 0$. If we replace $0 \leq a_n$ by $-1 \leq a_n$, can we conclude anything?
(3) Show that a bounded sequence in $\mathbb{R}$ need not converge.
(4) Show that a bounded sequence in $\mathbb{R}$ has a convergent subsequence.
(5) Let $\{x_n\}$ be a sequence in a topological space X. Define $y_n = x_{n+1}$ if n is odd and $y_n = x_{n-1}$ if n is even. Show that $\{x_n\}$ converges if and only if $\{y_n\}$ converges.
(6) Give an example of a sequence in a topological space which converges to more than one point.
(7) Let $\{x_n\}$ be a convergent sequence in a topological space and $k \in \mathbb{N}$. Show that if we replace $x_1, x_2, \ldots, x_k$ with points of our choice from X, then the resulting sequence will converge to the same point(s). Also, show that if we drop first k terms and start a new sequence from x_{k+1}, then also the resulting sequence will converge to the same point(s).
(8) Let $\{x_n\}$ be a Cauchy sequence in a metric space and $k \in \mathbb{N}$. Show that if we replace $x_1, x_2, \ldots, x_k$ with points of our choice from X, then the resulting sequence will be Cauchy too. Also, show that if we drop first k terms and start a new sequence from x_{k+1}, then the resulting sequence will be Cauchy too.

180. Let $\{x_n\}, \{y_n\}$ be Cauchy sequences in a metric space (X, d). Is $\{d(x_n, y_n)\}$ a Cauchy sequence in $\mathbb{R}$?

181. Let $\{x_n\}, \{y_n\}$ be two sequences in a metric space (X, d) such that $x_n \to x$ and $y_n \to y$ in X. Show that $d(x_n, y_n) \to d(x, y)$.

182. Let (X, d) be a metric space $\mathscr{T}_d$ be the topology induced by the metric, $\{x_n\}$ be a sequence in X and $x \in X$. Show that $x_n \to x$ in (X, d) if and only if $x_n \to x$ in $(X, \mathscr{T}_d)$.

27. Closed Subsets of a Metric Space

27.1. Proposition. *A closed ball in a metric space is a closed subset.*

Proof. Let (X, d) be a metric space, $x \in X$ and $r > 0$. Let $y \notin B[x, r]$. Then $t = d(x, y) > r$. Suppose, if possible, that $z \in B[x, r] \cap B(y, t - r)$. Then

$$t = d(x, y) \leq d(x, z) + d(z, y) < r + t - r = t,$$

a contradiction. Thus the neighbourhood $B(y, t - s)$ does not contain any point of $B[x, r]$. That is, $y \notin B[x, r]'$. So no point outside $B[x, r]$ is a limit point of $B[x, r]$. This means, $B[x, r]' \subset B[x, r]$. Thus $B[x, r]$ is closed. □

27.2. Example. Let (X, d) be a metric space, $x \in X$ and $r > 0$. Then

$$\overline{B(x, r)} \subset B[x, r].$$

However, the equality need not hold.

Since $B[x,r]$ is a closed set containing $B(x,r)$, and since $\overline{B(x,r)}$ is the smallest closed set containing $B(x,r)$, $\overline{B(x,r)} \subset B[x,r]$. Now consider a discrete metric space X with at least two distinct points x, y. Then

$$B(x,1) = \{x\} = B(x,0.5) \subset \overline{B(x,0.5)} \subset B[x,0.5] = \{x\} \neq X = B[x,1].$$

Hence $\overline{B(x,1)} = \{x\} \neq B[x,1]$.

The above example shows that the closure of an open ball need not be the corresponding closed ball. The following example shows that closure of an open ball need not be even a ball.

27.3. Example. Consider the metric $d(x,y) = \min\{|x-y|, 1\}$, $(x, y \in \mathbb{R})$. Then $B(0,1) = (-1,1)$, $\overline{B(0,1)} = [-1,1]$. However $[-1,1]$ is not a closed or open ball. Suppose $[-1,1] = B[x,r]$ for some $x \in \mathbb{R}$ and $r > 0$. Since the distance of 1 and 0 is 1, $r \geq 1$. But then $B[x,r] = \mathbb{R}$. Thus $[-1,1]$ is not a ball at all.

27.4. Proposition. *A finite subset of a metric space is closed.*

Proof. Since a finite set is a finite union of singletons, it is sufficient to prove that every singleton is closed in a metric space. So, let X be a metric space and $x \in X$. We show that $\{x\}' = \emptyset$. Clearly, x is not a limit point of $\{x\}$ because no neighbourhood of x contains a point of $\{x\}$ other than x. Now let $y \in X$ such that $x \neq y$. Let $r = d(x,y)$. Then $b(y, \frac{r}{2}) \cap \{x\} = \emptyset$. Thus $\{x\}' = \emptyset$. Hence $\{x\}$ is closed. □

The following characterizes closure of a set in a metric space using sequences.

27.5. Lemma. (**Sequence Lemma**) *Let X be a topological space, $A \subset X$. If there is a sequence $\{x_n\}$ in A converging to $x \in X$, then $x \in \overline{A}$. The converse holds if X is metrizable.*

Proof. Suppose there is a sequence $\{x_n\}$ in A such that $x_n \to x$. Let U be a neighbourhood of x. So, there is $N \in \mathbb{N}$ such that $x_n \in U$ for every $n \geq N$. Thus U contains points of A. Hence $x \in \overline{A}$.

Conversely, suppose there is a metric d such that the topology of X is given by d. Assume that $x \in \overline{A}$. We observe that for each $n \in \mathbb{N}$, $B(x, \frac{1}{n}) \cap A \neq \emptyset$. So, choose $x_n \in B(x, \frac{1}{n}) \cap A$ for each n. Thus $\{x_n\}$ is a sequence in A. Given an $\varepsilon > 0$, fix $N \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$. That is, $\frac{1}{N} < \varepsilon$. Then for $x_n \in B(x, \frac{1}{n}) \Rightarrow d(x_n, x) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ whenever $n \geq N$. Thus $x_n \to x$. □

The following example shows that the Sequence Lemma does not hold in an arbitrary topological space. At the same time, let us mention here that metric space in the Sequence Lemma can be replaced by more general spaces called *sequential spaces*. At present, we are not dealing with those spaces.

27.6. Example. Let X be an uncountable set with a cocountable topology. Fix $x \in X$ and let $A = X \setminus \{x\}$. Clearly, A is not closed as it is an uncountable proper subset of X. Hence $\overline{A} = X$. Thus $x \in \overline{A}$. But by the Example 26.23, a convergent

sequence from A will be eventually constant, say $x_1, x_2, \ldots, x_k, y, y, y, \ldots$ and will converge to $y \in A$. Of course, it will not converge to x because $X \setminus \{y\}$ is a neighbourhood of x that does not contain a tail of $\{x_n\}$. Thus $x_n \not\to x$.

The following corollary shows that the topology of a metric space is determined by its convergent sequences.

27.7. Corollary. *Let d, ρ be two metrics on a set X. The following are equivalent.*

(1) $\mathscr{T}_d = \mathscr{T}_\rho$.
(2) $x_n \to x$ with respect to d if and only if $x_n \to x$ with respect to ρ.

PROOF. ☞(1) ⇒ (2) is obvious.

☞(2) ⇒ (1) Assume (2). Let $G \in \mathscr{T}_d$. So, $A = X \setminus G$ is $\mathscr{T}_d$-closed. Let $x \in \overline{A}$ with respect to $\mathscr{T}_\rho$. So, by the Sequence Lemma, there is a sequence $\{x_n\}$ in A such that $x_n \to x$ in the metric ρ, and hence, in d. But then $x \in \overline{A}$ with respect to $\mathscr{T}_d$. Hence $x \in A$. Thus A is closed in $\mathscr{T}_\rho$. That is, $G \in \mathscr{T}_\rho$. The symmetry of the argument shows that $\mathscr{T}_\rho \subset \mathscr{T}_d$. □

Of course this is not true in arbitrary topological spaces. As Examples 26.21 and 26.23 show that the discrete and the cocountable topological spaces on a set X have the same convergent sequences. However, if X is uncountable, then these two topologies are different.

27.8. Proposition. *The metrics d_p, $(1 \le p \le \infty)$ on $\mathbb{R}^k$ are all equivalent to each other.*

PROOF. Let $1 \le p < \infty$, $k \in \mathbb{N}$ and $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k), \mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k) \in \mathbb{R}^k$. Then for every $1 \le j \le k$,

$$|\mathbf{x}_j - \mathbf{y}_j|^p \le \sum_{i=1}^{k} |\mathbf{x}_i - \mathbf{y}_i|^p \Rightarrow |\mathbf{x}_j - \mathbf{x}_j| < (\sum_{i=1}^{k} |\mathbf{x}_i - \mathbf{y}_i|^p)^{1/p} = d_p(\mathbf{x}, \mathbf{y}).$$

As a consequence,

$$\mathbf{x}_n \to \mathbf{x} \text{ in } (\mathbb{R}^k, d_p) \Rightarrow \mathbf{x}_{nj} \to \mathbf{x}_j \text{ in } \mathbb{R}$$

for every $1 \le j \le k$. On the other hand,

$$\begin{aligned}
\mathbf{x}_{nj} \to \mathbf{x}_j \text{ in } \mathbb{R} \text{ for all } j &\Rightarrow |\mathbf{x}_{nj} - \mathbf{x}_j| \to 0 \text{ for all } j \\
&\Rightarrow |\mathbf{x}_{nj} - \mathbf{x}_j|^p \to 0 \text{ for all } j \\
&\Rightarrow \sum_{j=1}^{k} |\mathbf{x}_{nj} - \mathbf{x}_j|^p \to 0 \\
&\Rightarrow (\sum_{j=1}^{k} |\mathbf{x}_{nj} - \mathbf{x}_j|^p)^{1/p} \to 0 \\
&\Rightarrow d_p(\mathbf{x}_n, \mathbf{x}) \to 0 \\
&\Rightarrow \mathbf{x}_n \to \mathbf{x} \text{ in } (\mathbb{R}^k, d_p).
\end{aligned}$$

Also for $p = \infty$,

$$|\mathbf{x}_{nj} - \mathbf{x}_j| < \max_{1\le i\le k} |\mathbf{x}_{ni} - \mathbf{y}_i|.$$

As a consequence, $\mathbf{x}_n \to \mathbf{x}$ in $(\mathbb{R}^k, d_\infty) \Rightarrow \mathbf{x}_{nj} \to \mathbf{x}_j$ in $\mathbb{R}$ for every $1 \le j \le k$. On the other hand, $\mathbf{x}_{nj} \to \mathbf{x}_j$ in $\mathbb{R}$ for every j, implies $d_\infty(\mathbf{x}_n, \mathbf{x}) \le d_1(\mathbf{x}_n, \mathbf{x}) \to 0$. Thus $\mathbf{x}_n \to \mathbf{x}$ in $(\mathbb{R}^k, d_p)$ for any p if and only if $\mathbf{x}_{nj} \to \mathbf{x}_j$. So, each of $(\mathbb{R}^k, d_p)$, $(1 \le p \le \infty)$ have the same convergent sequences. Consequently, all of them are equivalent. □

27.9. Proposition. *For a metric space (X, d), let $\bar{d}(x, y) = \min\{d(x, y), 1\}$, $(x, y \in X)$, be as in (25.10.1). Then d and $\bar{d}$ are equivalent.*

Proof. Let $\varepsilon > 0$ be given. Let $\varepsilon' = \min\{\varepsilon, 1\}$. Then $d(x, y) < \varepsilon' \Leftrightarrow \bar{d}(x, y) < \varepsilon'$. In both cases the required quantity will be less than ε. Thus a sequence converges with respect to d if and only if it converges with respect to $\bar{d}$. This completes the proof. □

In passing, we obtain yet another characterization of the closure of a set. We shall need the following definition.

27.10. Definition. Let (X, d) be a metric space, $x \in X$ and $A, B \subset X$. We define the *distance of x from A* to be

$$d(x, A) = \inf\{d(x, a) : a \in A\}. \tag{27.10.1}$$

We define the distance between A, B to be

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}. \tag{27.10.2}$$

27.11. Proposition. *Let X be a metric space, $x \in X$ and $A \subset X$. Then the following are equivalent. $x \in \overline{A}$ if and only if $d(x, A) = 0$.*

Proof. Suppose $x \in \overline{A}$. Let there be a sequence $\{x_n\}$ in A such that $x_n \to x$. Then $0 \le d(x, A) \le d(x, x_n) \to 0$. Consequently, $d(x, A) = 0$.

Conversely, assume that $d(x, A) = 0$. For any open set G containing x, we get $\varepsilon > 0$ such that $B(x, \varepsilon) \subset G$. But then $\inf\{d(x, a) : a \in A\} = d(x, A) = 0 < \varepsilon$. So, ε is not a lower bound of the set $\{d(x, a) : a \in A\}$. Hence there exists $y \in A$ such that $d(x, y) < \varepsilon$. Thus $y \in B(x, \varepsilon) \subset G$. Hence $A \cap G \ne \emptyset$, giving $x \in \overline{A}$. □

It is tempting to infer that for two sets A, B in a metric space (X, d), $d(A, B) = 0$ if and only if $\overline{A} \cap \overline{B} \ne \emptyset$. One way is true. If $x \in \overline{A} \cap \overline{B}$, then $d(A, B) = 0$. However, the other way is not true. The following example shows the same. It also shows that the analogue of the above proposition need not hold for two closed sets.

27.12. Example. There exist two disjoint closed sets A, B such that $d(A, B) = 0$.

Indeed, let $A = \{n + \frac{1}{n} : n \in \mathbb{N} \setminus \{1\}\}$ and $B = \mathbb{N}$. Then $d(A, B) \leq d(n + \frac{1}{n}, n) = \frac{1}{n} \to 0$. However, $A \cap B = \emptyset$.

27.13. Proposition. *Let (X, d) be a metric space and $A, B \subset X$ be such that $\overline{A} \cap \overline{B} \neq \emptyset$. Then $d(A, B) = 0$.*

Proof. Let $x \in \overline{A} \cap \overline{B}$ and $\varepsilon > 0$. Choose $a \in A, b \in B$ such that $d(a, x) < \varepsilon$ and $d(b, x) < \varepsilon$. Then $d(A, B) \leq d(a, b) \leq d(a, x) + d(b, x) < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $d(A, B) = 0$. □

The following is an example of a perfect set. This set is also called a *Cantor ternary set*

27.14. Example. Let $I_0 = I = [0, 1]$. In what follows, the expansion of numbers are in ternary system.

$$\begin{aligned}
I_1 &= [\tfrac{0}{3}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{3}{3}] \\
&= [0, 0.1] \cup [0.2, 1] \\
&= [0, 0.0\dot{2}] \cup [0.2, 0.\dot{2}] \\
I_2 &= [\tfrac{0}{3^2}, \tfrac{1}{3^2}] \cup [\tfrac{2}{3^2}, \tfrac{3}{3^2}] \cup [\tfrac{6}{3^2}, \tfrac{7}{3^2}] \cup [\tfrac{8}{3^2}, \tfrac{9}{3^2}] \\
&= [0, \tfrac{1}{3^2}] \cup [\tfrac{2}{3^2}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{2}{3} + \tfrac{1}{3^2}] \cup [\tfrac{2}{3} + \tfrac{2}{3^2}, 1] \\
&= [0, 0.00\dot{2}] \cup [0.02, 0.0\dot{2}] \cup [0.2, 0.20\dot{2}] \cup [0.22, 0.\dot{2}] \\
I_3 &= [\tfrac{0}{3^3}, \tfrac{1}{3^3}] \cup [\tfrac{2}{3^3}, \tfrac{3}{3^3}] \cup [\tfrac{6}{3^3}, \tfrac{7}{3^3}] \cup [\tfrac{8}{3^3}, \tfrac{9}{3^3}] \cup [\tfrac{18}{3^3}, \tfrac{19}{3^3}] \cup [\tfrac{20}{3^3}, \tfrac{21}{3^3}] \cup [\tfrac{24}{3^3}, \tfrac{25}{3^3}] \cup [\tfrac{26}{3^3}, \tfrac{27}{3^3}] \\
&= \cdots \quad \cdots \quad \cdots \quad \cdots
\end{aligned}$$

Define $C = \cap_{i=1}^{\infty} I_i$. It is known[1] that $x \in C$ if and only if the ternary expansion of x, $(x = 0.a_1a_2a_3a_4\ldots_3)$, consists of $0, 2$ $(a_i \in \{0, 2\})$. We prove that C is a perfect set.

Let $x = 0.a_1a_2a_3a_4\ldots_3 \in C$. Let $\varepsilon > 0$. Find $k > 0$ such that $\varepsilon > \frac{1}{3^k}$. Let y be obtained by changing a_{k+1} and only a_{k+1} in the ternary expansion of x. If a_{k+1} is 0, replace it with 2 and vice versa. Then the new number $y \in C$ and $|x - y| < \varepsilon$.

Now let $x = 0.a_1a_2a_3a_4\ldots_3 \notin C$. So, there is some k such that $a_k = 1$. Let $\varepsilon = \frac{1}{3^{k+1}}$ then $d(x, y) > \varepsilon$ for every $y \in C$.[2] Thus $x \notin \overline{C}$, that is, $x \notin C'$. This completes the proof.

Problems

183. Let X be a metric space and $A \subset X$. If $x \in A'$, then show that every neighbourhood of x contains infinitely many points of A. Hence conclude that

(1) A finite set in a metric space has no limit point.
(2) A' is closed.

Let X be a topological space in which every singleton is closed. Then show that both of the above hold in X. Also show that in an arbitrary topological space, the result is not true.

[1] Prove!
[2] Prove!

184. Let (X, d) be a metric space, $\emptyset \neq A \subset X$ and $x \in X$. Show that there is a sequence $\{x_n\}$ in A such that $d(x_n, x) \to d(x, A)$.

185. Let (X, d) be a metric space and A, B be nonempty subsets of X, Show that there are sequences $\{x_n\}$ in A and $\{y_n\}$ in B such that $d(x_n, y_n) \to d(A, B)$.

186. Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X.

(1) If $x_n \to x$, then show that $\{x_n\} \cup \{x\}$ is closed in X.
(2) If $x_n \to x$, then show that $\{x_n\}$ is closed in X if and only if $x = x_n$ for some $n \in \mathbb{N}$.
(3) If $\{x_n\}$ is Cauchy that does not converge in X, then show that $\{x_n\}$ is closed.

28. Continuous Functions in a Metric Space

In this section, we define and analyse the concept of a continuous function defined on a metric space. We start with the classical definition of a real-valued continuous function defined on $\mathbb{R}$ which is a part of the school curricula.

28.1. Definition. Let $a \in \mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous at a if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon. \tag{28.1.1}$$

If f is continuous at each $a \in \mathbb{R}$, then f is said to be continuous on $\mathbb{R}$.

28.2. Example. $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x$, $(x \in \mathbb{R})$, is continuous on $\mathbb{R}$.

Indeed for every $\varepsilon > 0$, take $\delta = \varepsilon$.[3] Then for $a \in \mathbb{R}$,

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon,$$

proving the continuity of f at a. Since $a \in \mathbb{R}$ is arbitrary, we conclude that f is continuous on $\mathbb{R}$.

Using the classical definition of continuity of a function on $\mathbb{R}$, we extend the concept of continuity of a function on a metric space.

28.3. Definition. Let $(X, d), (Y, \rho)$ be two metric spaces and $a \in \mathbb{R}$. A function $f : X \to Y$ is said to be continuous at a if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$d(x, a) < \delta \Rightarrow \rho(f(x), f(a)) < \varepsilon. \tag{28.3.1}$$

If f is continuous at each $a \in X$, then f is said to be continuous on X.

It is clear from the metric on $\mathbb{R}$, that the Definition 28.3 reduces to the Definition 28.1 for a function $f : \mathbb{R} \to \mathbb{R}$. Now we prove that the definition of continuity of a function between two metric spaces is the same as the definition of continuity of a function between two topological spaces.

[3] Note that the choice of δ does not depend upon any point. It depends only on the given ε.

28.4. Theorem. *Let $(X,d),(y,\rho)$ be metric spaces and $f\colon X\to Y$ be a function. For $a\in X$, the following are equivalent.*

(1) f is continuous at a.
(2) For every neighbourhood V of $f(a)$, there exists a neighbourhood U of a such that $f(U)\subset V$.

PROOF. ☞(1) ⇒ (2) Let V be a neighbourhood of $f(a)$. Hence there exists $\varepsilon>0$, such that $B(f(a),\varepsilon)\subset V$. By the continuity of f, we get $\delta>0$ such that $d(x,a)<\delta\Rightarrow\rho(f(x),f(a))<\varepsilon$. Let $U=B(a,\delta)$. Then

$$x\in U\Rightarrow d(x,a)<\delta\Rightarrow\rho(f(x),f(a))<\varepsilon\Rightarrow f(x)\in V.$$

Thus $f(U)\subset V$.

☞(2) ⇒ (1) Let $\varepsilon>0$ be given. Let $V=B(f(a),\varepsilon)$. Then by the hypothesis, there is an open set $U\subset X$ such that $x\in U$ and $f(U)\subset V$. But then there is $\delta>0$ such that $B(a,\delta)\subset U$. Thus

$$d(x,a)<\delta\Rightarrow x\in U\Rightarrow f(x)\in V=B(f(a),\varepsilon)\Rightarrow\rho(f(x),f(a))<\varepsilon.$$

Thus f is continuous. □

Now we characterize continuity of a function on a metric space using sequences.

28.5. Theorem. *Let $(X,d),(Y,\rho)$ be metric spaces, $x\in X$ and $f\colon X\to Y$ be a function. The following are equivalent.*

(1) f is continuous at x.
(2) $x_n\to x$ in $X\Rightarrow f(x_n)\to f(x)$.

PROOF. Suppose f is continuous at x. Let $x_n\to x$ in X. Let $\varepsilon>0$. By the continuity of f, there is $\delta>0$ such that $d(x,y)<\delta\Rightarrow\rho(f(x)),f(y)<\varepsilon$. But then there exists $N\in\mathbb{N}$ such that $d(x_n,x)<\delta$ for every $n\geq N$. That, in turn, gives, $\rho(f(x_n),f(x))<\varepsilon$ for $n\geq N$. Thus $f(x_n)\to f(x)$.

Conversely, if f is discontinuous at x, then there is $\varepsilon>0$ such that for ever $n\in\mathbb{N}$, there is $x_n\in X$ such that $d(x_n,x)<\frac{1}{n}$ but $\rho(f(x_n),f(x))\geq\varepsilon$. Thus $x_n\to x$ but $f(x_n)\not\to f(x)$, a contradiction. So, f must be continuous at x. □

28.6. Theorem. *The following functions are continuous.*

(1) $f(\mathbf{x}_1,\mathbf{x}_2)=\mathbf{x}_1+\mathbf{x}_2$, $(\mathbf{x}_1,\mathbf{x}_2)\in\mathbb{R}^2$.
(2) $g(\mathbf{x}_1,\mathbf{x}_2)=\mathbf{x}_1\mathbf{x}_2$, $(\mathbf{x}_1,\mathbf{x}_2)\in\mathbb{R}^2$.
(3) $h(\mathbf{x}_1,\mathbf{x}_2)=\mathbf{x}_1-\mathbf{x}_2$, $(\mathbf{x}_1,\mathbf{x}_2)\in\mathbb{R}^2$.
(4) $k(\mathbf{x}_1,\mathbf{x}_2)\in\mathbb{R}^2=\frac{\mathbf{x}_1}{\mathbf{x}_2}$, $(\mathbf{x}_1,\mathbf{x}_2)\in\mathbb{R}^2\setminus\{(\mathbf{x}_1,0):\mathbf{x}_1\in\mathbb{R}\}$.

FIGURE 27. Continuity of +

PROOF. These have already been proved by you in your analysis course. We just demonstrate the proof of two of them.

☞(1) Let $a = (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^2$, $\varepsilon > 0$, $c = \mathbf{x}_1 + \mathbf{x}_2$ and $V = (c - \varepsilon, c + \varepsilon)$. Let $\delta = \frac{\varepsilon}{2}$, $U = (\mathbf{x}_1 - \delta, \mathbf{x}_1 + \delta) \times (\mathbf{x}_2 - \delta, \mathbf{x}_2 + \delta)$. Then $(\mathbf{y}_1, \mathbf{y}_2) \in U \Rightarrow \mathbf{x}_1 - \delta < \mathbf{y}_1 < \mathbf{x}_1 + \delta; \mathbf{x}_2 - \delta < \mathbf{y}_2 < \mathbf{x}_2 + \delta \Rightarrow c - \varepsilon < \mathbf{y}_1 + \mathbf{y}_2 < c + \varepsilon$. Thus f is continuous at a, and hence, on $\mathbb{R}^2$.

We give analytic proof of the same which will help us in proving the continuity of other functions.

$$|f(\mathbf{x}_1, \mathbf{x}_2) - f(\mathbf{y}_1, \mathbf{y}_2)| = |\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{y}_1 - \mathbf{y}_2| \le |\mathbf{x}_1 - \mathbf{y}_1| + |\mathbf{x}_2 - \mathbf{y}_2| < \varepsilon,$$

whenever $|\mathbf{x}_1 - \mathbf{y}_1| < \frac{\varepsilon}{2}$ and $|\mathbf{x}_2 - \mathbf{y}_2| < \frac{\varepsilon}{2}$.

☞(2)
$$\begin{aligned}(\mathbf{x}_{1n}, \mathbf{x}_{2n}) \to (\mathbf{x}_1, \mathbf{x}_2) &\Rightarrow \mathbf{x}_{1n} \to \mathbf{x}_1, \mathbf{x}_{2n} \to \mathbf{x}_2 \\ &\Rightarrow \mathbf{x}_{1n}\mathbf{x}_{2n} \to \mathbf{x}_1\mathbf{x}_2 \\ &\Rightarrow g(\mathbf{x}_{1n}, \mathbf{x}_{2n}) \to g(\mathbf{x}_1, \mathbf{x}_2).\end{aligned}$$

Thus g is continuous. □

28.7. Corollary. *Let Y be a topological space and $f, g : Y \to \mathbb{R}$ be functions. Then the following functions are continuous. (1) $f + g$, (2) $f - g$, (3) fg and (4) $\frac{f}{g}$, wherever defined.*

28.8. Definition. Let $(X, d), (Y, \rho)$ be metric spaces and $x \in X$. A function $f : X \to Y$ is said to be *uniformly continuous* if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon. \tag{28.8.1}$$

28.9. Proposition. *Let (X, d) be a metric space, $\emptyset \neq A \subset X$, $x \in X$ and $f : X \to \mathbb{R}$ be defined by $f(x) = d(x, A)$, $(x \in X)$. Then f is uniformly continuous.*

PROOF. Let $a \in A$. Then $d(x, A) \le d(x, a) \le d(x, y) + d(y, a)$ for every $x, y \in X$. That is, $d(x, A) \le d(x, y) + d(y, a)$. Now taking infimum over $a \in A$, and changing the side of $d(y, A)$, we get, $d(x, A) - d(y, A) \le d(x, y)$. The symmetry in x, y gives $d(y, A) - d(x, A) \le d(x, y)$, and hence,

$$|f(x) - f(y)| = |d(x, A) - d(y, A)| \le d(x, y). \tag{28.9.1}$$

Now given $\varepsilon > 0$, let $\delta = \varepsilon$.

$$d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

Hence f is uniformly continuous. □

The inequality (28.9.1) suggest a special type of functions.

28.10. Definition. Let (X, d) and (Y, ρ) be two metric spaces. A function $f\colon X \to Y$ is said to be a Lipschitz function if there exists $M > 0$ such that for all $x, y \in X$

$$\rho(f(x), f(y)) \leq Kd(x, y). \tag{28.10.1}$$

The following is at once.

28.11. Proposition. *A Lipschitz function is uniformly continuous.*

28.12. Theorem. *Let $(X, d), (Y, \rho)$ be metric spaces and $f\colon X \to Y$ be a uniformly continuous function. Then f maps Cauchy sequences to Cauchy sequences.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in X and $\varepsilon > 0$. Then there is $\delta > 0$ such that $d(x, y) < \delta \Rightarrow \rho(f(x), f(x)) < \varepsilon$. So, there is $N \in \mathbb{N}$ such that $d(x_n, x_m) \leq \delta$ for all $n, m \geq M$. Compiling all this, we get

$$n, m \geq N \Rightarrow d(x_n, x_m) < \delta \Rightarrow \rho(f(x_n), f(x_m)) < \varepsilon.$$

Thus $\{f(x_n)\}$ is Cauchy. □

In general, a continuous function need not send Cauchy sequences to Cauchy sequences. The easy example is $f\colon (0, \infty) \to (0, \infty)$ defined by $f(x) = \frac{1}{x}$, $(x \in (0, \infty))$. It maps a Cauchy sequence $\{\frac{1}{n}\}$ to $\{n\}$, which is not Cauchy.

28.13. Definition. Let $(X, d), (Y, \rho)$ be metric spaces, $x \in X$. A function $f\colon X \to Y$ is said to be an *isometry* if $\rho(f(x), f(y)) = d(x, y)$ for every $x, y \in X$. If f is an onto isometry, then we say that *X is isometric to Y*.

Clearly, an isometry is a Lipschitz function, and hence, uniformly continuous.

28.14. Proposition. *An onto isometry is a homeomorphism.*

Proof. Clearly, $d(x, y) < \varepsilon \Rightarrow \rho(f(x), f(y)) < \varepsilon$. Thus f as well as f^{-1} are continuous. So, the result follows. □

Isometries preserve a number of properties, which otherwise are not preserved by a homeomorphism.

28.15. Proposition.

(1) Isometric image of a Cauchy sequence is a Cauchy sequence.
(2) Isometric image of a convergent sequence is a convergent sequence.

PROOF. Let $(X,d),(Y,\rho)$ be metric spaces and $f\colon X\to Y$ be an isometry.

☞(1) Let $\{x_n\}$ be a Cauchy sequence in X. Given an $\varepsilon>0$, there is $N\in\mathbb{N}$ such that $d(x_n,x_m)<\varepsilon$ whenever $n,m\geq N$. But then $\rho(f(x_n),f(x_m))=d(x_n,x_m)<\varepsilon$ whenever $n,m\geq N$.

☞(2) Let $\{x_n\}$ be a convergent sequence in X. Let $x_n\to x$. Given an $\varepsilon>0$, there is $N\in\mathbb{N}$ such that $d(x_n,x)<\varepsilon$ whenever $n\geq N$. But then $\rho(f(x_n),f(x))=d(x_n,x)<\varepsilon$ whenever $n\geq N$. Thus $f(x_n)\to f(x)$. □

Problems

187. Using your school mathematics, show that the function $f\colon\mathbb{R}\to\mathbb{R}$ defined by each of the following formulas is continuous on $\mathbb{R}$.

(1) $-x$, (2) x^2, (3) $\sin x$ and (4) $\cos x$

188. Let $f\colon(x,d)\to(Y,\rho)$ be a function. Show that f is continuous at $a\in X$ if it satisfies the following.

Given $\varepsilon>0$, there is $\delta>0$ such that

$$d(x,a)\leq\delta\Rightarrow\rho(f(x),f(a))\leq\varepsilon.$$

189. Let $f\colon(x,d)\to(Y,\rho)$ be a function. Show that f is continuous at $a\in X$ if it satisfies the following.

Given $\varepsilon>0$, there is $\delta>0$ such that

$$d(x,a)\leq\delta\Rightarrow\rho(f(x),f(a))\leq 2\varepsilon.$$

190. Construct more examples of continuous functions f which do not map Cauchy sequences to Cauchy sequences.

191. Let $(X,d),(Y,\rho)$ be metric spaces, $x\in X$ and $f\colon X\to Y$ be a function. Show that each of the following is equivalent to the statement "f is continuous at x".

(1) $x\in\mathrm{Int}(f^{-1}(B(f(x),r)))$ for every $r>0$.
(2) $x\in\mathrm{Int}(f^{-1}(B(f(x),r)))$ for every $0<r<1$.
(3) $x\in\mathrm{Int}(f^{-1}(B(f(x),r)))$ for every $r\in\mathbb{Q}$.
(4) $x\in\mathrm{Int}(f^{-1}(B(f(x),\frac{1}{n})))$ for every $n\in\mathbb{N}$.
(5) $x\in\mathrm{Int}(f^{-1}(B[f(x),r]))$ for every $r>0$.
(6) $x\in\mathrm{Int}(f^{-1}(B[f(x),r]))$ for every $0<r<1$.
(7) $x\in\mathrm{Int}(f^{-1}(B[f(x),r]))$ for every $r\in\mathbb{Q}$.
(8) $x\in\mathrm{Int}(f^{-1}(B[f(x),\frac{1}{n}]))$ for every $n\in\mathbb{N}$.

29. Bounded and Totally Bounded Subsets

29.1. Definition. Let (X,d) be a metric space and $A\subset X$. We define the *diameter of A* to be

$$\mathrm{diam}(A)=\sup\{d(x,y):x,y\in A\}. \tag{29.1.1}$$

We say that A is *bounded* if $\mathrm{diam}(A)<\infty$. A set which is not bounded is called an *unbounded*.

29.2. Remark.

(1) Recall that $\sup\emptyset = -\infty$ and $\inf\emptyset = \infty$. So, $\operatorname{diam}(\emptyset) = -\infty$, and hence, $\emptyset$ is bounded.
(2) Boundedness of the subsets of a metric space depend upon the underlying metric. If we replace a metric by an equivalent metric, then boundedness of a subset may be affected. For example, $\mathbb{R}$ with the usual metric is unbounded but with $\rho(x,y) = \frac{|x-y|}{1+|x-y|}$, $(x, y \in \mathbb{R})$, every subset of $\mathbb{R}$ is bounded.
(3) It is clear that a sequence in a metric space is bounded if and only if its range is a bounded set.

29.3. Proposition. *The following subsets of a metric space are bounded.*

(1) Subset of a bounded set.
(2) A finite union of bounded sets.
(3) A finite set.
(4) The range of a Cauchy sequence.
(5) The range of a convergent sequence.
(6) A closed ball.
(7) An open ball.

Proof. Let (X, d) be a metric space.

☞(1) Let $A \subset X$ be bounded and $B \subset A$. Then

$$\operatorname{diam}(B) = \sup\{d(a,b) : a, b \in B\} \leq \sup\{d(a,b) : a, b \in A\} < \infty.$$

Hence B is bounded.

☞(2) If we prove that union of two bounded sets is bounded, the result follows by repeating the argument finite number of times. So, let $A, B \subset X$ be bounded. Fix $x \in A$ and $y \in B$. Let $a, b \in A \cup B$.

$$\begin{aligned} a, b \in A \Rightarrow d(a,b) &\leq \operatorname{diam}(A) \\ &\leq \operatorname{diam}(A) + d(x,y) + \operatorname{diam}(B). \\ a, b \in B \Rightarrow d(a,b) &\leq \operatorname{diam}(B) \\ &\leq \operatorname{diam}(A) + d(x,y) + \operatorname{diam}(B). \\ a \in A, b \in B \Rightarrow d(a,b) &\leq d(a,x) + d(x,y) + d(y,b) \\ &\leq \operatorname{diam}(A) + d(x,y) + \operatorname{diam}(B). \end{aligned}$$

Taking supremum over all $a, b \in A \cup B$, we get,

$$\operatorname{diam}(A \cup B) \leq \operatorname{diam}(A) + d(x,y) + \operatorname{diam}(B)$$

☞(3) follows from (2) and the fact that diameter of a singleton is 0.

☞(4) Let $\{x_n\}$ be Cauchy. Let $N \in \mathbb{N}$ be such that $d(x_n, x_m) \leq 1$ for $n, m \geq N$. This gives boundedness of $A = \{x_n : n \geq N\}$ by restricting its diameter by 1. But then the range of $\{x_n\}$ being union of the finite set $\{x_n : n < N\}$ and a bounded set A, is bounded.

☞(5) follows by observing that a convergent sequence is Cauchy.

☞(6) Diameter of $B[x, r]$ is at most $2r$, and hence, it is bounded.

☞(7) follows as an open ball is subset of the corresponding closed ball. □

The following characterizes the bounded sets.

29.4. Proposition. *Let (X, d) be a metric space, $A \subset X$ and $a \in X$. The following are equivalent.*

(1) A is bounded.
(2) $A \subset B(x, r)$ for some $x \in X$ and some $r > 0$.
(3) $A \subset B[x, r]$ for some $x \in X$ and some $r > 0$.
(4) $A \subset B(a, t)$ for some $t > 0$.
(5) $A \subset B[a, t]$ for some $t > 0$.

Proof. ☞(1)⇒(2) Let $\operatorname{diam}(A) \leq s \in (0, \infty)$. Then for $x, y \in A$, $d(x, y) < s + 1 = r$, say. Thus $A \subset B(x, r)$.

☞(2)⇒(3) follows from $A \subset B(x, r) \subset B[x, r]$.

☞(3)⇒(4) Let $t = r + d(x, a) + 1$. For $y \in A$, $d(a, y) \leq d(a, x) + d(x, y) \leq d(a, x) + r < t$. Thus $A \subset B(a, t)$.

☞(4)⇒(5) follows from $A \subset B(a, t) \subset B[a, t]$.

☞(5)⇒(1) $\operatorname{diam}(A) \leq \operatorname{diam}(B[a, t]) \leq 2t < \infty$. So, A is bounded. □

29.5. Proposition. *Closure of a bounded set is bounded.*

Proof. Let A be a bounded subset of a metric space (X, d). So, there is a closed ball $B[x, r]$ such that $A \subset B[x, r]$. But then $\overline{A} \subset B[x, r]$, and hence, A is bounded. □

29.6. Definition. Let (X, d) be a metric space. $A \subset X$ is said to be *totally bounded* if for every $\varepsilon > 0$, there is a finite set $\{x_1, x_2, x_3, \ldots, x_n\} \subset X$ such that $A \subset \cup_{i=1}^{n} B(x_i, \varepsilon)$. In this case, the finite set $\{x_1, x_2, x_3, \ldots, x_n\}$ is called an *ε-net* for A.

29.7. Proposition. *The following subsets of a metric space are totally bounded.*

(1) Subset of a totally bounded set.
(2) A finite union of totally bounded sets.
(3) A finite set.
(4) The range of a Cauchy sequence.
(5) The range of a convergent sequence.
(6) Closure of a totally bounded set.

Proof. Let (X, d) be a metric space.

☞(1) Suppose $B \subset A \subset X$ and $\varepsilon > 0$. Then there is $\{x_1, x_2, x_3, \ldots, x_n\} \subset X$ such that $A \subset \cup_{i=1}^{n} B(x_i, \varepsilon)$. But then $B \subset A \subset \cup_{i=1}^{n} B(x_i, \varepsilon)$.

☞(2) If $\{A_k : 1 \leq k \leq n\}$ is a finite family of totally bounded subsets of X, and given an $\varepsilon > 0$, if $A_k \subset \cup_{i=1}^{r_k} B(x_{k,i}, \varepsilon)$, then $\cup_{k=1}^{n} A_k \subset \cup_{k=1}^{n} (\cup_{i=1}^{r_k} B(x_{k,i}, \varepsilon))$. This proves (2).

☞(3) A finite set is a finite union of singletons, and hence, by (2), proof follows.

☞(4) Let $\{x_n\}$ be a Cauchy sequence. Given an $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $T_N(\{x_n\}) \subset B(x_N, \varepsilon)$. The remaining terms $\{x_1, x_2, x_3, \ldots, x_{N-1}\}$ of the sequence can be covered by at most $N - 1$ open ε-spheres.

☞(5) follows by observing that a convergent sequence is Cauchy.

☞(6) Let A be a totally bounded subset of X and $\varepsilon > 0$ be given. So, there exists a finite set $\{x_1, x_2, x_3, \ldots, x_n\} \subset X$ such that $A \subset \cup_{i=1}^{n} B(x_i, \frac{\varepsilon}{2})$. But then $\overline{A} \subset \cup_{i=1}^{n} \overline{B(x_i, \frac{\varepsilon}{2})} \subset \cup_{i=1}^{n} B[x_i, \frac{\varepsilon}{2}] \subset \cup_{i=1}^{n} B(x_i, \varepsilon)$. □

29.8. Example. A bounded set, need not be totally bounded. Indeed, $\mathbb{R}$ with the discrete metric space is bounded but is not totally bounded as we do not get finitely many open balls of radius $\frac{1}{2}$ to cover $\mathbb{R}$. Putting the same in different words, $\mathbb{R} = B(0, 2)$ is an open sphere which is not totally bounded.

30. Complete Metric Spaces

30.1. Definition. A metric space (X, d) is said to be *complete* if every Cauchy sequence in X is also convergent.

You have studied that $\mathbb{R}, \mathbb{R}^n$ with the usual metrics are complete metric spaces. We shall use the fact.

30.2. Proposition. *$\mathbb{R}^k$ with each metric d_p, $(1 \leq p \leq \infty)$ is complete.*

Proof. Let $\{\mathbf{x}_n\}$ be a Cauchy sequence in $\mathbb{R}^k$. As noted in the Proposition 27.8

$$|\mathbf{x}_j - \mathbf{y}_j| \leq d_p(\mathbf{x}, \mathbf{y}).$$

Consequently, $\{\mathbf{x}_{nj}\}$ is Cauchy in $\mathbb{R}$ for $1 \leq j \leq k$. Since $\mathbb{R}$ is complete, $\{\mathbf{x}_{nj}\}$ converges in $\mathbb{R}$. Now appealing to the Proposition 27.8, we conclude that $\{\mathbf{x}_n\}$ converges in $\mathbb{R}^k$. □

30.3. Proposition. *Let X be a complete metric space and Y be subspace of X. Then Y is complete if and only if Y is closed in X.*

Proof. Let Y be complete and $x \in \overline{Y}$. Then there exists a sequence $\{x_n\}$ in Y such that $x_n \to x$. Since every convergent sequence is Cauchy, so is $\{x_n\}$. But

then $\{x_n\}$ is a Cauchy sequence in a complete metric space Y; and hence, $\{x_n\}$ converges in Y, say $x_n \to y$. But then $x_n \to y$ as a sequence in X also. Since a convergent sequence has a unique limit, $x = y$. This gives $x \in Y$ because $y \in Y$. Thus $\overline{Y} \subset Y$, asserting the closedness of y.

Conversely, suppose Y is closed. Let $\{x_n\}$ be a Cauchy sequence in Y. Then $\{x_n\}$, as a sequence in X, is Cauchy; and hence, by the completeness of X, is convergent. Say $x_n \to x$. But then we have a sequence $\{x_n\}$ in Y converging to x. Since Y is closed, $x \in Y$. Thus $\{x_n\}$ converges in Y. So, Y is complete. □

The following is a generalization of the Nested Interval Theorem.

30.4. Theorem. (Cantor's Intersection Theorem) *Let (X, d) be a complete metric space and $\{F_n\}_{n=1}^{\infty}$ be a collection of subsets of X such that*

(1) Each F_n is closed.
(2) $F_1 \supset F_2 \supset \cdots$.
(3) $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$.

Then $\cap_{n=1}^{\infty} F_n$ is a singleton.

Proof. Suppose some $F_k = \emptyset$. Also $F_k \supset F_{k+1} \supset F_{k+2} \cdots \Rightarrow F_n = \emptyset$ for all $n \geq k$. So, $\operatorname{diam}(F_n) = -\infty$ for all $n \geq k$. Hence $\operatorname{diam}(F_n) \not\to 0$, a contradiction to (3). Thus no F_k is empty. For each $n \in \mathbb{N}$, choose $x_n \in F_n$ and also fix $\varepsilon > 0$. Then by (3), there exists $N \in \mathbb{N}$ such that $\operatorname{diam}(F_n) < \varepsilon$ for all $n \geq N$. Let $n, m > N$. Without loss of generality we assume that $n < m$. Hence $x_m \in F_m \subset F_n$. But $x_n \in F_n$. Hence $d(x_n, x_m) \leq \operatorname{diam}(F_n) < \varepsilon$. Thus $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$. Hence $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, we get $x \in X$ such that $x_n \to x$. Now fix $i \in \mathbb{N}$. Then Clearly, $\{x_{n+i}\}$ is a sequence in F_i and $x_{n+i} \to x$ as $n \to \infty$. Hence $x \in \overline{F_i} = F_i$. Hence $x \in F_i$ for every $i \in \mathbb{N}$. Hence $x \in \cap_{n=1}^{\infty} F_n$. Finally, suppose there is another $y \neq x$ such that $y \in \cap_{n=1}^{\infty} F_n$. But then $d(x, y) \leq \operatorname{diam}(F_n) \to 0$, and hence, $d(x, y) = 0$. Thus $x = y$, and so, $\cap_{n=1}^{\infty} F_n = \{x\}$, a singleton. This completes the proof. □

We shall use Cantor's Intersection Theorem to prove the following theorem known as the Baire's Category Theorem.

30.5. Theorem. (Baire's Category Theorem) *A complete metric space is of second category. That is, in other words, if $\{A_n\}$ is a sequence of nowhere dense sets in a complete metric space (X, d), then $\cup_{n=1}^{\infty} A_n \neq X$. That is, there exists a point x outside the set $\cup_{n=1}^{\infty} A_n$.*

Proof. Let (X, d) be a complete metric space and $\{A_n\}$ be a sequence of nowhere dense sets in X. Suppose every nonempty open set U intersects A_1. But then $\overline{A_1} = X$, giving $\emptyset = (\overline{A_1})^{\circ} = X$, a contradiction. So, there is an open set $U_0 \subset X$ such that $A_1 \cap U_0 = \emptyset$. Let $B_1 = B(x_1, r_1)$ be an open ball of radius $r_1 < 1$ such that $B_1 \subset U_0$. That is, $B_1 \cap A_1 = \emptyset$. Let $F_1 = B[x_1, \frac{r_1}{2}]$. Now $A_2^{\circ} = \emptyset$. So, choose an open ball $B_2 = B(x_2, r_2) \subset F_1^{\circ}$ of radius $r_2 < \frac{1}{2}$ such that $B_2 \cap A_2 = \emptyset$. Define $F_2 = B[x_2, \frac{r_2}{2}]$. Continuing in this way, for every $n \in \mathbb{N}$, we get the open balls

$B_n = B(x_n, r_n) \subset F_{n-1}^\circ$ with $r_n < \frac{1}{n}$ and $B_n \cap A_n = \emptyset$. Define $F_n = B[x_n, \frac{r_n}{2}]$. Thus for every $n \in \mathbb{N}$, F_n is a closed set, $F_n \subset F_{n-1}$ and $\operatorname{diam}(F_n) \leq 2\frac{r_n}{2} = r_n < \frac{1}{n}$. Hence by the Cantor's Intersection Theorem, $\cap_{n=1}^\infty F_n = \{x\}$, some singleton. Clearly, for any $i \in \mathbb{N}$, $x \in \cap_{n=1}^\infty F_n \subset F_i$ and $F_i \cap A_i = \emptyset$. So, $x \notin A_i$. Thus $x \notin \cup_{n=1}^\infty A_n$. Thus $\cup_{n=1}^\infty A_n \neq X$. Hence X is of second category. □

30.6. Theorem. *Let (X, d) be a metric space and $Y \subset X$ be dense in X. If every Cauchy sequence in Y converges in X, then every Cauchy sequence in X also converge in X. Consequently, X is complete.*

PROOF. Let $\{x_n\}$ be a Cauchy sequence in X. Since Y is dense in X, there is a sequence $\{y_n\}$ in Y such that $d(x_n, y_n) < \frac{1}{n}$.

☞$\{y_n\}$ is Cauchy.

Let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that $d(x_n, x_m) \leq \frac{\varepsilon}{3}$, whenever $n, m \geq N$. Now let $M \in \mathbb{N}$ such that $\frac{1}{M} < \frac{\varepsilon}{3}$. Then for $n, m \geq N + M$,

$$d(y_n, y_m) \leq d(y_n, x_n) + d(x_n, x_m) + d(x_m, y_m) < \frac{1}{M} + \frac{\varepsilon}{3} + \frac{1}{M} < \varepsilon.$$

Now by the hypothesis, $\{y_n\}$ converges in X. Say $y_n \to x$ in X. We show that $x_n \to x$. Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\varepsilon}{2}$ and $M > 0$ such that $d(y_n, x) < \frac{\varepsilon}{2}$, whenever $n \geq M$. Now for $n \geq N + M$,

$$d(x_n, x) \leq d(x_n, y_n) + d(y_n, x) < \frac{1}{n} + \frac{\varepsilon}{2} \leq \varepsilon.$$

Thus $x_n \to x$. □

30.7. Theorem. *Let (X, d) be a metric space, $Z \subset X$, and (Y, ρ) be a complete metric space and $f : Z \to Y$ be a uniformly continuous map. Then f has a unique continuous extension to $\overline{Z}$.*

PROOF. Let $x \in \overline{Z}$. By the Sequence Lemma (Lemma 27.5), there is $x_n \in Z$ such that $x_n \to x$. So, $\{x_n\}$ is Cauchy, and hence, by the Theorem 28.12, $\{f(x_n)\}$ is Cauchy in Y. Since Y is complete, $\{f(x_n)\}$ is convergent in Y. Define $g(x) = \lim_{n\to\infty} f(x_n)$.

☞g is well-defined.

Let $x \in \overline{Z}$. Suppose $\{x_n\}, \{y_n\}$ be two sequences from Z both converging to x. Suppose $f(x_n) \to a$, $f(y_n) \to b$. Let $\varepsilon > 0$ be given. Choose $M_1, M_2 \in \mathbb{N}$ such that

$$n \geq M_1 \Rightarrow \rho(f(x_n), a) < \frac{\varepsilon}{3} \quad \text{and} \quad n \geq M_1 \Rightarrow \rho(f(y_n), b) < \frac{\varepsilon}{3}.$$

Also choose $\delta > 0$ such that $d(u, v) < \delta \Rightarrow \rho(f(u), f(v)) < \frac{\varepsilon}{3}$. Let $N_1, N_2 \in \mathbb{N}$ be such that $d(x_n, x) < \frac{\delta}{2}$, whenever $n \geq N_1$ and $d(y_n, x) < \frac{\delta}{2}$ whenever $n \geq N_2$. Let $M = N_1 + N_2 + M_1 + M_2$. Then

$$d(x_M, y_M) \leq d(x_M, x) + d(x, y_M) < \delta.$$

Consequently,

$$\rho(a,b) \leq \rho(a, f(x_M)) + \rho(f(x_M), f(y_M)) + \rho(f(y_M), b) < \varepsilon.$$

Thus g is well-defined.

☞g is continuous.

Let $\varepsilon > 0$. Choose $\delta > 0$ such that $x, y \in Z, d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \frac{\varepsilon}{2}$. Let $x, y \in \overline{Z}$ and $d(x,y) < \frac{\delta}{2}$. Let $\{x_n\}, \{y_n\}$ be sequences from Z such that $x_n \to x$, $y_n \to y$. Choose $N \in \mathbb{N}$ such that $d(x_n, x) \leq \frac{\delta}{4}$, $d(y_n, y) < \frac{\delta}{4}$ whenever $n \geq N$. Then

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) < \delta.$$

Consequently, $\rho(f(x_n), f(y_n)) < \frac{\varepsilon}{2}$, and hence, taking limit, we have

$$\rho(g(x), g(y)) = \lim_{n\to\infty} \rho(f(x_n), f(y_n)) \leq \frac{\varepsilon}{2} < \varepsilon.$$

☞Uniqueness of g.

Suppose $h : \overline{Z} \to Y$ is another continuous extension of f. Let $x \in \overline{Z}$ and $\{x_n\}$ be a sequence from Z such that $x_n \to x$. Then

$$h(x) = \lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} f(x_n) = g(x).$$

This completes the proof. □

30.8. Corollary. *Let (X,d) be a metric space, $Z \subset X$, (Y,ρ) be a complete metric space and $f : Z \to Y$ be an isometry. Then the unique continuous extension of f to $\overline{Z}$ is an isometry.*

Proof. Let g be the unique extension of f to $\overline{Z}$. Let $x, y \in \overline{Z}$ and $\{x_n\}, \{y_n\}$ be sequences from Z such that $x_n \to x$ and $y_n \to y$. But then $\rho(g(x), g(y)) = \lim_{n\to\infty} \rho(f(x_n), f(y_n)) = \lim_{n\to\infty} d(x_n, y_n) = d(x,y)$. □

Problems

192. Show that $\mathbb{Z}$ is the complete.

193. Show that a discrete metric space is the complete.

194. Show that $(0,1)$ is not complete but it is homeomorphic to $\mathbb{R}$.

195. Show that on $\mathbb{Q}$ there is no complete metric equivalent to the usual metric.

196. Find the intersection of F_n in each of the following. Also, find which property of the hypothesis of the Cantor's Intersection Theorem does not hold in each case.

(1) $F_n = [n, \infty) \subset \mathbb{R}$, $(n \in \mathbb{N})$.
(2) $F_n - (0, \frac{1}{n}) \subset \mathbb{R}$, $(n \in \mathbb{N})$.
(3) $F_n - (0, \frac{1}{n}] \subset (0,1)$, $(n \in \mathbb{N})$.

197. Consider $F_n = (-\frac{1}{n}, \frac{1}{n}) \subset (-2, 2)$, $n \in \mathbb{N}$. Observe that F_n is not closed, $F_n \supset F_{n+1}$ for all $n \in \mathbb{N}$, and $\operatorname{diam}(F_n) \to 0$ as $n \to \infty$. What is the $\cap_{n=1}^{\infty}$? Does this situation clash with the Cantor's Intersection Theorem?

31. Completion of a Metric Space

A number of results do not hold for incomplete metric spaces. Thanks to the tool we are going to develop in this section guaranteeing that every metric space can be regarded as a dense subset of a complete metric space uniquely up to isometry.

31.1. Lemma. *Let (X,d) be a metric space. Let $\mathscr{C}$ denote the collection of all Cauchy sequences in (X,d). For $\{x_n\},\{y_n\} \in \mathscr{C}$, define $\{x_n\} \sim \{y_n\}$ if $d(x_n,y_n) \to 0$. Then $\sim$ is an equivalence relation.*

Proof. It is easy to see that the relation $\sim$ is reflexive and symmetric. We prove the transitivity. Let $\{x_n\} \sim \{y_n\}$ and $\{y_n\} \sim \{z_n\}$. Then

$$0 \le d(x_n,z_n) \le d(x_n,y_n) + d(y_n,z_n) \to 0.$$

Thus $d(x_n,z_n) \to 0$, which completes the proof. □

We shall denote the equivalence class containing the Cauchy sequence $\{x_n\}$ under the equivalence relation $\sim$ defined in the Lemma 31.1 by $[x_n]$.

31.2. Lemma. *Let (X,d) and $\sim$ be as in the Lemma 31.1. Define*

$$X^* = \{[x_n] : \{x_n\} \in \mathscr{C}\}, \tag{31.2.1}$$

Also, for $[x_n],[y_n] \in X^*$, define

$$\rho([x_n],[y_n]) = \lim_{n\to\infty} d(x_n,y_n). \tag{31.2.2}$$

Then ρ is a metric on X^*.

Proof. ☞ $\lim_{n\to\infty} d(x_n,y_n)$ exists.

Let $[x_n],[y_n] \in X^*$ and $\varepsilon > 0$. Then there are N_1, N_2 such that

$$d(x_n,x_m) < \frac{\varepsilon}{2} \text{ whenever } n,m \ge N_1 \text{ and } d(y_n,y_m) < \frac{\varepsilon}{2} \text{ whenever } n,m \ge N_2.$$

Consequently, for $n,m \ge N_3 = \max\{N_1,N_2\}$,

$$\begin{aligned} d(x_n,y_n) - d(x_m,y_m) &\le d(x_n,x_m) + d(x_m,y_n) - d(x_m,y_m) \\ &\le d(x_n,x_m) + d(x_m,y_m) + d(y_m,y_n) - d(x_m,y_m) \\ &= d(x_n,x_m) + d(y_m,y_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

By the symmetry of the argument in n and m, we have

$$|d(x_n,y_n) - d(y_m,y_m)| < \varepsilon,$$

whenever $n,m \ge N_3$. Thus $\{d(x_n,y_n)\}$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, we conclude that $\{d(x_n,x_m)\}$ converges in $\mathbb{R}$.

☞ ρ is well-defined.

Let $[x_n], [y_n], [z_n] \in X^*$ and $[x_n] \sim [y_n]$. Then

$$\begin{aligned}\rho([x_n],[z_n]) &= \lim_{n\to\infty} d(x_n, z_n)\\ &\le \lim_{n\to\infty}(d(x_n,y_n)+d(y_n,z_n))\\ &= \lim_{n\to\infty} d(y_n,z_n)\\ &= \rho([y_n],[z_n]).\end{aligned}$$

Similarly,

$$\rho([y_n],[z_n]) \le \rho([x_n],[z_n]).$$

Thus ρ is well-defined.

☞ρ is positive definite and symmetric. This sit-n-do part is left to the reader.

☞Triangle inequality for ρ. Let $[x_n], [y_n], [z_n] \in X^*$.

$$\begin{aligned}\rho([x_n],[y_n]) &= \lim_{n\to\infty} d(x_n, y_n)\\ &\le \lim_{n\to\infty} d(x_n,z_n)+d(z_n,y_n)\\ &= \rho([x_n],[z_n])+\rho([z_n],[y_n]).\end{aligned}$$

□

31.3. Lemma. *Let $(X,d), X^*$ be as in the Lemma 31.2. For $x \in X$, let $\tilde{x}$ denote the constant sequence $\{x,x,x,\dots\}$. Define $f\colon X \to X^*$ by $f(x) = [\tilde{x}]$, $(x \in X)$. Then f is an isometry and $f(X)$ is dense in X^*.*

Proof. Let $x, y \in X$. Define $x_n = x$, $y_n = y$ for all $n \in \mathbb{N}$. Then $\rho(f(x), f(y)) = \rho([\tilde{x}],[\tilde{y}]) = \lim_{n\to\infty} d(x,y) = d(x,y)$. Thus f is an isometry.

Let $[x_n] \in X^*$. Let $\varepsilon > 0$. Since $\{x_n\}$ is a Cauchy sequence in X, there is $N \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ for every $n, m > N$. Let $z = x_N$. Then $d(x_n, z) < \frac{\varepsilon}{2}$ for every $n \ge N$. Hence $\rho([x_n],[\tilde{z}]) = \lim_{n\to\infty} d(x_n, x_N) \le \frac{\varepsilon}{2} < \varepsilon$. Thus $[\tilde{z}] \in B([x_n], \varepsilon) \cap f(X)$. Thus $f(X)$ is dense in X^*. □

31.4. Lemma. *Let $(X,d), X^*$ as in the Lemma 31.3. Then X^* is complete.*

Proof. In view of the Theorem 30.6, it is sufficient to show that every Cauchy sequence in $f(X)$ converges in X. Let $\{[\tilde{x_k}]\}$ be a Cauchy sequence in $f(X)$. We show that $[\tilde{x_k}] \to [x_n]$.

Note

To realize the sequence, observe that $\tilde{x_1}$ is the constant sequence $x_1, x_1, x_1, \dots$, which is Cauchy, and hence, $[\tilde{x_1}] \in X^*$ which is the first element of the sequence $\{[\tilde{x_k}]\}$. In the same way, $\tilde{x_2}$ is the constant sequence $x_2, x_2, x_2, \dots$, which is Cauchy, and hence, $[\tilde{x_2}] \in X^*$. Continuing in this way $\tilde{x_k}$ is the constant sequence $x_k, x_k, x_k, \dots$, which is Cauchy, and hence, $[\tilde{x_k}] \in X^*$. On the other hand, $[x_n]$ is the equivalence class of the Cauchy sequence $\{x_1, x_2, x_3, \dots\}$.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that $d(x_n, x_m) < \frac{\varepsilon}{2}$, whenever $n, m \geq N$. Now let $k \geq N$. Then

$$\rho([\tilde{x_k}], [x_n]) = \lim_{n\to\infty} d(x_k, x_n) \leq \frac{\varepsilon}{2} < \varepsilon.$$

Thus $[\tilde{x_k}] \to [x_n]$ as $k \to \infty$. □

Summing up all these Lemmas, we have the following.

31.5. Theorem. *Let (X, d) be a metric space. There is a unique, up to isomorphism, complete metric space (X^*, ρ) containing isometrically X as a dense subset.*

Proof. In views of the above series of lemmas, we have

(1) (X^*, ρ) is a complete metric space.
(2) $f\colon (X, d) \to (X^*, \rho)$ is an isometry.
(3) $f(X)$ is dense in X.

It remains to prove that

(4) If (Y, σ) is a metric space and $h\colon (X, d) \to (Y, \sigma)$ is an isometry with a dense range, then there is an onto isometry $k\colon (Y, \sigma) \to (X^*, \rho)$.

Note that $f \circ h^{-1}\colon h(X) \to X^*$ is an isometry. So, it extends uniquely as a continuous map $k\colon Y \to X^*$ to the closure Y of $h(X)$. Since the isometric image of a complete metric space is complete, $k(Y)$ is complete. But $f(X) = f \circ h^{-1}(h(X)) \subset k(h(X)) \subset k(Y) \subset X^*$, it follows that $h(Y)$ is a dense as well as closed subset of X^*. Hence $k(Y) = X^*$. Since k is a continuous extension of an isometry, k is isometry. Thus Y is isometric to X^*. This completes the proof. □

Problems

198. Show that $\mathbb{R}$ is the completion of $\mathbb{Q}$.

199. Show that $\mathbb{R}$ is the completion of $\mathbb{R} \setminus \mathbb{Q}$.

200. Let $(X_1, d_1), (X_2, d_2)$ be metric spaces and (X_1^*, d_1) and (X_2^*, d_2) denote their completion respectively. Show that $(X_1^* \times X_2^*, d)$ is the completion of $(X_1 \times X_2, d)$, where $d((\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2)) = \max\{d_1(\mathbf{x}_1, \mathbf{y}_1), d_2(\mathbf{x}_2, \mathbf{y}_2)\}$, $((\mathbf{x}_1, \mathbf{x}_2), (\mathbf{y}_1, \mathbf{y}_2) \in X_1 \times X_2)$.

CHAPTER 6

Convergence

In general, sequences are not sufficient in arbitrary topological spaces to characterize the topology and the continuity of functions. So, now we generalize the notion of a sequence to the object known as a *net*. Nets and their convergence are not new to you. In fact, you have used it in Riemann Integration. Let $f\colon [a,b] \to \mathbb{R}$ be a bounded function. The upper Riemann sums of f over $[a,b]$ form a net and it does converge. In the same way, The lower Riemann sums of f over $[a,b]$ form a net too and it does converge too. If both of them converge to the same number, then the number is called the Riemann integral of f.

32. Inadequacy of Sequences

Let us recall that the Sequence Lemma (Lemma 27.5) characterizes closure of a set and the Theorem 28.5 characterizes the continuity of a function in the set up of metric spaces. In fact, such a result is true in more general set up of *sequential spaces*. However, none of these holds true in arbitrary topological spaces. The following examples show the same.

Recall from the Example 26.23 that a sequence in a cocountable topological space converges if and only if it is eventually constant.

32.1. Example. Let $\mathbb{R}_c, \mathbb{R}_d$ denote the real line with the cocountable topology and the discrete topology respectively. Both have the same set of convergent sequences, viz. the eventually constant sequences.

(1) $[0,1]$ is dense in $\mathbb{R}_c$, and hence, $2 \in \overline{[0,1]}$ but there is no sequence from $[0,1]$ that converges to 2.
(2) $\iota\colon \mathbb{R}_c \to \mathbb{R}_d$ is discontinuous however $x_n \to x$ in $\mathbb{R}_c \Rightarrow \iota(x_n) \to \iota(x)$ in $\mathbb{R}_d$.
(3) Let $X = \mathbb{R}^{\mathbb{R}}$. Let $A = \{\mathbf{x} \in \mathbb{R}^{\mathbb{R}} : \mathbf{x} = \chi_F$, where $F \subset \mathbb{R}$ is finite$\}$. Then $\chi_{\mathbb{R}} \in \overline{A}$. But there is no sequence from A that converges to $\chi_{\mathbb{R}}$.

Problems

201. In a finite set with any topology, show that sequences are sufficient to characterize the closures of subsets.

33. Nets

33.1. Definition. A relation $\leq$ on a set $\Lambda \neq \emptyset$ is said to be a *direction* if
(1) $\leq$ is reflexive.
(2) $\leq$ is transitive.
(3) Given $\lambda, \mu \in \Lambda$, there is $\gamma \in \Lambda$ such that $\lambda \leq \gamma$ and $\mu \leq \gamma$.
Λ with $\leq$ is called a *directed set*

33.2. Definition. Let X be a set and Λ be a directed set. A function $\lambda \in \Lambda \mapsto x_\lambda \in X$ is called a *net*. A net is usually denoted by $\{x_\lambda\}_{\lambda\in\Lambda}$, and when there is no confusion, by $\{x_\lambda\}$. Let $\{x_\lambda\}$ be a net in X. For $\mu \in \Lambda$, the set $T_\mu(\{x_\lambda\}) = \{x_\lambda : \lambda \geq \mu\}$ is called a *tail of the net* $\{x_\lambda\}$.

33.3. Definition. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in a set X and $A \subset X$. We say that
(1) $\{x_\lambda\}$ is *eventually* in A if A contains a tail of $\{x_\lambda\}$.
(2) $\{x_\lambda\}$ is *frequently* in A if every tail of $\{x_\lambda\}$ intersects A.

33.4. Definition. For directed set Λ and Γ a function $\varphi\colon \Gamma \to \Lambda$ is said to be
(1) *increasing* if $\mu \leq \gamma \Rightarrow \varphi(\mu) \leq \varphi(\gamma)$.
(2) *cofinal* if given $\lambda \in \Lambda$, there is $\mu \in \Gamma$ such that $\lambda \leq \varphi(\mu)$.

33.5. Definition. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in a set X, Γ be a directed set and $\mu \in \Gamma \mapsto \varphi(\mu) \in \Lambda$ be an increasing cofinal map. Then the map $\mu \in \Gamma \mapsto x_{\varphi(\mu)} \in X$ is called a *subnet* of (x_λ) and is denoted by $\{x_{\varphi(\mu)}\}$.

33.6. Example.
(1) $\mathbb{N}$ with the usual order is a directed set, and hence, every sequence is a net.
(2) A nonempty subset of $\mathbb{R}$ with the usual order are a directed sets.
(3) Let X be a set. $\Lambda = \mathscr{P}(X)$ with relation $A \leq B$ if $A \subset B$ is a directed set. This is called the *upward direction*
(4) Each of the above with the reversed order is also a directed set. This is called the *downward direction* However, unlike the partial order, a reversed order of a directed set need not make it a directed set. See Exercise 202.
(5) Let X be a topological space and $x \in X$. Then $\Lambda = \mathscr{U}_x$ with $U \leq V$ whenever $V \subset U$ is a directed set. Whenever we consider $\mathscr{U}_x$ as a directed set, we shall mean this order, unless explicitly mentioned otherwise.

33.7. Definition. Let X be a set. On any collection $\mathscr{A}$ of subsets of X, define $E \leq F$ if $E \subset F$, $(E, F \in \mathscr{A})$. If $(\mathscr{A}, \leq)$ is a directed set, then we say that $\leq$ is an *upward direction*. Now, define $E \leq F$ if $F \subset E$, $(E, F \in \mathscr{A})$. If $(\mathscr{A}, \leq)$ is a directed set, then we say that $\leq$ is an *downward direction*.

33.8. Remark. Now onwards without mentioning the directed set Λ, we shall state that $\{x_\lambda\}$ is a net.

33.9. Definition. Let X be a topological space, $\{x_\lambda\}$ be a net in X. We say that $\{x_\lambda\}$ is *convergent* if there is $x \in X$ if every neighbourhood of x contains a tail of $\{x_\lambda\}$. In this case, we say that $\{x_\lambda\}$ *converges* to x and write $x_\lambda \to x$.

Important Note

Instead of $x_\lambda \to x$, never ever write $\lim_{\lambda\to\infty} x_\lambda = x$ or $x_\lambda \to x$ as $\lambda \to \infty$. This is meaningless. Think what happens when $\Lambda = \{1, 2, 3, 4\}$ with the usual order. However, writing $\lim_{\lambda} x_\lambda = x$ is perfectly permissible.

33.10. Lemma. *Let X be a topological space and $x \in X$. For each $U \in \mathscr{U}_x$, choose $x_U \in U$. Then the net $x_U \to x$.*

Proof. Let V be a neighbourhood of x. Then the tail $T_V(\{x_U\}) \subset V$. □

We recapture the closure of a set and continuity of a function through nets.

33.11. Theorem. *Let X be a topological space, $x \in X$ and $A \subset X$. Then $x \in \overline{A}$ if and only if there is a net from A converging to x.*

Proof. Assume that $x \in \overline{A}$. For each $U \in \mathscr{U}_x$, choose $x_U \in A \cap U$. Then $\{x_U\}$ is a net from A and by the Lemma 33.10, $x_U \to x$.

Conversely, let $\{x_\lambda\}$ be a net from A such that $x_\lambda \to x$. Let V be a neighbourhood of x. Since $x_\lambda \to x$, V contains a tail $T_\mu(\{x_\lambda\})$ of the net $\{x_\lambda\}$. But then $x_\mu \in A \cap U$. Thus $x \in \overline{A}$. □

33.12. Theorem. *Let X, Y be topological spaces, $x \in X$ and $f : X \to Y$ be a function. The following are equivalent.*

(1) f is continuous at x.
(2) If $x_\lambda \to x$ in X, then $f(x_\lambda) \to f(x)$ in Y.

Proof. Suppose f is continuous at x and $x_\lambda \to x$ in X. Let $V \in \mathscr{U}_{f(x)}$. Then $f^{-1}(V) \in \mathscr{U}_x$. Consequently, there is a tail $T_\mu(\{x_\lambda\}) \subset f^{-1}(V)$. Applying f both the sides, we get $T_\mu(\{f(x_\lambda)\}) \subset f(f^{-1}(V)) \subset V$. This guarantees that $f(x_\lambda) \to f(x)$.

Conversely, suppose f is discontinuous at x. So, for some $V \in \mathscr{U}_{f(x)}$, $f^{-1}(V) \notin \mathscr{U}_x$. Thus for every $U \in \mathscr{U}_x$, $U \not\subset f^{-1}(V)$. Consequently, for every $U \in \mathscr{U}_x$, there is $x_U \in U \setminus f^{-1}(V)$. But then by the Lemma 33.10, $x_U \to x$, but no term of $\{f(x_U)\}$ is in V. That is, $x_U \to x$ but $f(x_U) \not\to f(x)$. □

33.13. Definition. Let X be a topological space, $\{x_\lambda\}$ be a net in X. We say that $\{x_\lambda\}$ *clusters* at $x \in X$ if $\{x_\lambda\}$ is frequently in each neighbourhood of x.

33.14. Theorem. *Let $\{x_\lambda\}$ be a net in a topological space X and $x \in X$. The following are equivalent.*

(1) $\{x_\lambda\}$ clusters at x.
(2) There is a subnet of $\{x_\lambda\}$ which converges to x.

In particular, if $x_\lambda \to x$, then $\{x_\lambda\}$ clusters at x.

Proof. Let $\{x_\lambda\}_{\lambda\in\Lambda}$ cluster at x. Let $\Gamma = \{(U,\lambda) : x_\lambda \in U \in \mathscr{U}_x\}$ with the order $(\lambda, U) \le (\mu, V)$ whenever $\lambda \le \mu, V \subset U$. Let the function $\varphi\colon \Gamma \to \Lambda$ be defined by $f(\lambda, U) = \lambda$. Then $(\lambda, U) \in \Gamma \mapsto x_{\varphi(\lambda,U)} = x_\lambda \in X$ is a subnet of $\{x_\lambda\}$. Clearly, $(\lambda, U) \le (\mu, V) \Rightarrow \varphi(\lambda, U) = \lambda \le \mu = \varphi(\mu, V)$. Also, for $\lambda \in \Lambda$, $(\lambda, X) \in \Gamma$ and $\varphi(\lambda, X) = \lambda \ge \lambda$. Thus φ is cofinal. Now let $U \in \mathscr{U}_x$. Choose $\lambda \in \Lambda$ such that $x_\lambda \in U$. Let $(\mu, V) \ge (\lambda, U)$. Then $x_{\varphi(\mu,V)} = x_\mu \in V \subset U$. Thus $x_{\varphi(\lambda,U)} \to x$.

Conversely, suppose a subnet $\{x_{\varphi(\gamma)}\}_{\gamma\in\Gamma}$ of $\{x_\lambda\}_{\lambda\in\Lambda}$ converges to x. Let $U \in \mathscr{U}_x$ and $\mu \in \Lambda$. Choose $\gamma_0 \in \Gamma$ such that $T_{\gamma_0}\{(x_{\varphi(\gamma)})\} \subset U$. Let $\delta \in \Gamma$ be such that $\varphi(\delta) \ge \mu$. Let $\eta \in \Gamma$ be such that $\delta \le \eta$ and $\gamma_0 \le \eta$. Then $\varphi(\eta) \ge \mu$ and $x_{\varphi(\eta)} \in U$. Thus $\{x_\lambda\}$ is frequently in U. Thus $\{x_\lambda\}$ clusters at x. □

33.15. Definition. A net $\{x_\lambda\}$ in a set X is said to be an *ultranet* if for every $E \subset X$, $\{x_\lambda\}$ is eventually either in E or in $X \setminus E$.

33.16. Remark. An eventually constant net is an ultranet however an explicit example of an ultranet which is not eventually constant is difficult. We shall prove the existence of such an ultranet implicitly.

33.17. Proposition. *Let $\{x_\lambda\}$ be an ultranet in a set X and $E \subset X$. If $\{x_\lambda\}$ is frequently in E, then it is eventually in E.*

Proof. Suppose $\{x_\lambda\}$ is eventually in $X \setminus E$. Then $X \setminus E$ contains a tail of $\{x_\lambda\}$, say $T_\mu(\{x_\lambda\})$. But then there is no $\delta \ge \mu$ such that $x_\delta \in E$. Thus $\{x_\lambda\}$ is not frequently in E, a contradiction. Thus $\{x_\lambda\}$ must be eventually in E. □

33.18. Corollary. *Let $\{x_\lambda\}$ be an ultranet in a topological space X. If $\{x_\lambda\}$ clusters at $x \in X$, then $x_\lambda \to x$.*

33.19. Theorem. *Image of an ultranet is an ultranet.*

PROOF. Let X, Y be two sets, $f\colon X \to Y$ be a function and $\{x_\lambda\}$ be an ultranet in X. Clearly for $E \subset Y$, $x_\lambda \in f^{-1}(E) \Leftrightarrow f(x_\lambda) \in f(f^{-1}(E)) \subset E$. Thus $\{f(x_\lambda)\}$ is an ultranet. □

The following is apparent.

33.20. Proposition. *Let $\{x_n\}$ be a sequence in a topological space X and $x \in X$. $x_n \to x$ as a sequence if and only if $x_n \to x$ as a net.*

33.21. Theorem. *Let $\{(X_i, \mathscr{T}_i) : i \in I\}$ be a family of topological spaces and $\mathbf{x} \in X = \prod_{i\in I} X_i$, the product space. A net $\{\mathbf{x}_\lambda\}$ in X converges to $\mathbf{x}$ if and only if $\pi_i(\mathbf{x}_\lambda) \to \pi_i(\mathbf{x})$ for each $i \in I$.*

PROOF. One way implication is a consequence of the continuity of the projections. On the other hand, assume that $\pi_i(\mathbf{x}_\lambda) \to \pi_i(\mathbf{x})$ for each $i \in I$. Let $S = \pi_j^{-1}(U_j)$ be a subbasic neighbourhood of $\mathbf{x}$ with $U_j \in \mathscr{T}_j$. Clearly, $\mathbf{x}_j \in U_j$, and hence, for some $\mu \in \Lambda$, $T_\mu(\{\mathbf{x}_{\lambda j}\}) \subset U_j$. Consequently, $T_\mu(\{\mathbf{x}_\lambda\}) \subset \pi_j^{-1}(U_j) = S$. Now for $B = S_1 \cap S_2 \cap S_3 \cap \cdots \cap S_n$ be a basic neighbourhood of $\mathbf{x}$. Then there are $\mu_1, \mu_2, \mu_3, \ldots, \mu_n$ such that $T_{\mu_k}(\{\mathbf{x}_\lambda\}) \subset S_k$ for $1 \le k \le n$. Let $\mu \in \Lambda$ be such that $\mu_k \le \mu$ for $1 \le k \le n$. Then $T_\mu(\{\mathbf{x}_\lambda\}) \subset B$. Since every $U \in \mathscr{U}_\mathbf{x}$ contains some basic neighbourhood B of $\mathbf{x}$, it follows that $\mathbf{x}_\lambda \to \mathbf{x}$. □

33.22. Theorem. *Let $\{(X_i, \mathscr{T}_i) : i \in I\}$ be a family of topological spaces and $\mathbf{x} \in X = \prod_{i\in I} X_i$, the product space. If a net $\{\mathbf{x}_\lambda\}$ in X clusters at $\mathbf{x}$, then $\{\pi_i(\mathbf{x}_\lambda)\}$ clusters at $\pi_i(\mathbf{x}_i)$ for each $i \in I$.*

PROOF. Assume that $\{\mathbf{x}_\lambda\}$ clusters at $\mathbf{x}$. Let $U_i \in \mathscr{T}_i$ be a neighbourhood of $\mathbf{x}_i$ and $\mu \in \Lambda$. Then there is $\lambda \ge \mu$ such that $\mathbf{x}_\lambda \in \pi_i^{-1}(U_i)$. But then $\pi_i(\mathbf{x}_\lambda) \in U_i$. Thus $\{\pi_i(\mathbf{x}_\lambda)\}$ clusters at $\mathbf{x}_i$. □

The converse of the above theorem does not hold even in case of $\mathbb{R} \times \mathbb{R}$. The following example shows it.

33.23. Example. Define

$$\mathbf{x}_n = \begin{cases} (1, n), & \text{if } n \text{ is odd;} \\ (n, 2), & \text{if } n \text{ is even.} \end{cases}$$

Then $\{\pi_1(\mathbf{x}_n)\}$ clusters at 1 and $\{\pi_2(\mathbf{x}_n)\}$ clusters at 2 but $\{\mathbf{x}_n\}$ does not cluster at any point.

In passing, we state that the concept of Cauchy nets is not defined in arbitrary topological spaces. However, we can surely define a Cauchy net in a metric space and a uniform space.

33.24. Definition. Let (X, d) be a metric space and $\{x_\lambda\}$ be a net in X. We say that $\{x_\lambda\}$ is *Cauchy* if for every $\varepsilon > 0$, there is a λ_0 such that $d(x_\lambda, x_\mu) < \varepsilon$ whenever $\lambda, \mu \geq \lambda_0$.

In fact, we shall deal with more general spaces, called *uniform spaces*, than metric spaces in which we shall define the concept of Cauchy nets (see the Definition 56.1). Naturally, more general results proved in the context of uniform spaces will be applicable to metric spaces. So, we do not discuss results on Cauchy nets in metric spaces here.

Problems

202. Let X be a set. Let Λ be the collection of all nonempty finite subsets of X. Show that Λ is a directed set with the usual order of subsets, that is, $A \leq B$ if $A \subset B$. Show that Λ is not a directed set with the reverse order.

203. Show that a subnet of a sequence need not be a subsequence.

204. Show that tail of a net is a subnet but not conversely.

205. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let $\mathscr{P}$ be the collection of all partitions of $[a, b]$ and for $P \in \mathscr{P}$, let $U(P, f)$ and $L(P, f)$ denote the upper and lower Riemann sum over the partition $\mathscr{P}$. Show that $P \in \mathscr{P} \mapsto U(P, f)$ and $P \in \mathscr{P} \mapsto L(P, f)$ are nets in $\mathbb{R}$.

206. Let X be a set, $\Lambda = \{F \subset X : F \text{ is finite }\}$ is a directed set with upward direction, that is, $E \subset F$ if $E \subset F$, and $f : X \to \mathbb{R}$ be a function. For a finite set $F \subset X$, define $S_F = \sum_{x \in F} f(x)$. Show that $\{S_F\}_F$ is a net. The limit, if exists, is called the sum of f over X and is denoted by $\sum_{x \in X} f(x)$, a discrete version of the integration. Suppose that $f : X \to (0, \infty)$. Show that if the net $\{S_F\}_F$ converges, then it converges to $\sup\{S_F : F \subset X \text{ is finite }\}$.

207. Let X be an indiscrete topological space. Show that every net in X converges to all the points of X.

208. Show that a net in a discrete space converges if and only if it is eventually constant.

209. Let $\Lambda = \{(a, F) \subset \mathbb{R} : a \notin F \text{ and } F \text{ is countable}\}$. For $(a, E), (b, F) \in \Lambda$, define $(a, E) \leq (b, F)$ if $E \subset F$. Does the net $(a, F) \mapsto a \in \mathbb{R}$ converge to any point in $\mathbb{R}$ with the cocountable topology?

210. Which of the nets considered in the previous exercises cluster at some point?

211. Show that a subnet of an ultranet is an ultranet.

34. α-Nets and α-graded spaces

34.1. Definition. Let X be a set and $\{x_\lambda\}_{\lambda \in \Lambda}$ be a net. If $|\Lambda| = \alpha$, then we say that $\{x_\lambda\}$ is an *α-net*.

It should be noted that every α-net is a net.

34.2. Definition. Let α be a cardinal. We say that a topological space is *α-graded* if every $x \in X$ has a local base $\mathscr{B}_x$ with $|\mathscr{B}_x| \leq \alpha$. The first countable spaces are $\aleph_0$-graded.

Similar to the characterization of closures by sequences, we have the following in α-graded spaces.

34.3. Theorem. *Then α-nets describe the topology of an α-graded space X.*

PROOF. Let X be an α-graded space. Let $A \subset X$ and $x \in X$. Suppose $x \in \overline{A}$. Let $\mathscr{B}_x$ be a local base at x with $|\mathscr{B}_x| = \alpha$. Applying the axiom of choice, choose $x_B \in B$ for each $B \in \mathscr{B}_x$. Clearly, (x_B) is an α-net in A such that $x_B \to x$. The converse is obvious because every α-net is a net. Thus the closure of a set in an α-graded topological space is characterized by convergent α-nets. □

34.4. Theorem. *Let X be an α-graded topological space, Y be a topological space and $f : X \to Y$ be a function. Then the following are equivalent.*

(1) f is continuous at $x \in X$.
(2) $x_\lambda \to x$ in $X \Rightarrow f(x_\lambda) \to f(x)$ for every α-net (x_λ) in X.

PROOF. One way it known as every α-net is a net.

Conversely, suppose (2) holds. Suppose f is discontinuous at $x \in X$; so that there is a neighbourhood V of $f(x)$ in Y such that $f^{-1}(V)$ is not a neighbourhood of x. Let $\mathscr{B}_x$ be a local base at x in X such that $|\mathscr{B}_x| \leq \alpha$. Then for every $B \in \mathscr{B}_x$, there is $x_B \in B$ such that $x_B \notin f^{-1}(V)$. That is, $x_B \to x$ but $\{f(x_B)\} \cap V = \emptyset$. Thus $f(x_B) \not\to f(x)$, a contradiction. Thus f must be continuous. □

34.5. Corollary. *Homeomorphic image of an α-graded space is an α-graded space.*

34.6. Theorem. *A co α-topological space X is discrete if and only if $|X| \leq \alpha$.*

PROOF. Suppose $\mathscr{T}_\alpha$ is discrete. Let $x \in X$. Then $\{x\}$ is open, and hence, $|X \setminus \{x\}| \leq \alpha$. Hence $|X| = |X \setminus \{x\}| \leq \alpha$.

Conversely, if $|X| \leq \alpha$, then $|X \setminus \{x\}| \leq |X| \leq \alpha$ for every $x \in X$, asserting that $\{x\}$ is open, and hence, $\mathscr{T}_\alpha$ is discrete. □

34.7. Theorem. *A convergent α-net in a co α-topological space X is eventually constant.*

PROOF. Let $(x_\lambda)_{\lambda \in \Lambda}$ be an α-net. Suppose $x_\lambda \to x$. Let $G = (X \setminus \{x_\lambda : \lambda \in \Lambda\}) \cup \{x\}$. Then G is a neighbourhood of x that can contain a tail of the net (x_λ) only if the tail constants to x. □

34.8. Corollary. *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{T}_\alpha \leq \mathscr{T}$. Then an α-net in X converges if and only if it is eventually constant.*

Proof. Clearly, the identity map $\iota\colon (X, \mathscr{T}) \to (X, \mathscr{T}_\alpha)$ is continuous. So, the result follows by applying the Theorem 34.4. □

34.9. Corollary. *Let X be a co α-topological space and $|X| > \alpha$. Then X is not α-graded.*

Proof. Let $U = \{x\}$. Then $X \setminus \{x\}$ is not open because $|X \setminus \{x\}| = |X| > \alpha$. Thus $A = X \setminus \{x\}$ is not closed, so that $x \in \overline{A}$. But as we have seen any α-net from A converging to x must be eventually constant net with a tail consisting of x only. Thus there cannot be an α-net from A converging to x. So, X is not α-graded. □

Problems

212. Consider a cardinal topologies $\mathscr{T}_{<\alpha}$ and $\mathscr{T}_\alpha$. Let β be a cardinal such that $\beta \leq \alpha$. Is it true that all β-nets are essentially constant in $\mathscr{T}_{<\alpha}$ and $\mathscr{T}_\alpha$

35. Filters

35.1. Definition. Let X be a set. A nonempty family $\mathscr{F}$ of nonempty subsets of X is said to be a *filter* on X if

(1) $E, F \in \mathscr{F} \Rightarrow E \cap F \in \mathscr{F}$.
(2) $E \in \mathscr{F}, E \subset F \Rightarrow F \in \mathscr{F}$.

35.2. Theorem. *Let $\mathscr{F}$ be a filter on a set X. Then $\mathscr{T} = \mathscr{F} \cup \{\emptyset\}$ is a topology on X.*

Proof. First we show that $\mathscr{T}$ defined by $\mathscr{F} \cup \{\emptyset\}$ is a topology. Indeed, $\emptyset, X \in \mathscr{T}$. Now let $\{G_i : i \in I\} \subset \mathscr{T}$. Fix a nonempty G_j. Then $G_j \in \mathscr{F}$ and $G_j \subset \cup_{i\in I} G_i$. Hence, $\cup_{i\in I} G_i \in \mathscr{T}$. Let $G_1, G_2 \in \mathscr{T}$. If G_1 or G_2 is empty, then we are done. So, assume that $G_1, G_2 \in \mathscr{F}$. But then $G_1 \cap G_2 \in \mathscr{F}$. Thus $\mathscr{T}$ is a topology on X □

Of course, the converse to the above is not true. Indeed if we remove the empty set from the usual topology on $\mathbb{R}$, then the resulting family is not a filter on $\mathbb{R}$.

35.3. Definition. Let $\mathscr{F}, \mathscr{G}$ be filters on a set X. We say that $\mathscr{G}$ is a *finer filter* than $\mathscr{F}$ if $\mathscr{F} \subset \mathscr{G}$. In this case, we write $\mathscr{F} \leq \mathscr{G}$.

35.4. Definition. Let $\mathscr{F}$ be a filter on a set X. A family $\mathscr{C}$ is said to be a filter base for $\mathscr{F}$ if $\mathscr{F} = \{F \subset X : \text{there is } C \in \mathscr{C} \text{ such that } C \subset F\}$.

35.5. Proposition. *Let X be a nonempty set and $\mathscr{C}$ be a nonempty family of nonempty subsets of X. Then $\mathscr{C}$ is a filter base for some filter if and only if $C, D \in \mathscr{C} \Rightarrow$ there is $B \in \mathscr{C}$ such that $B \subset C \cap D$.*

Proof. Suppose $C, D \in \mathscr{C} \Rightarrow$ there is $B \in \mathscr{C}$ such that $B \subset C \cap D$. Let $\mathscr{F} = \{E \subset X :$ there is $C \in \mathscr{C}$ such that $C \subset F\}$.

Let $E, F \in \mathscr{F}$. So that there are $C, D \in \mathscr{C}$ such that $C \subset E, D \subset F$. Hence there is $B \in \mathscr{C}$ such that $B \subset C \cap D \subset E \cap F$. Thus $E \cap F \in \mathscr{F}$. Also, let $E \in \mathscr{F}$ and $E \subset F$. Then there is $C \in \mathscr{C}$ such that $C \subset E$. Hence, $C \subset E \subset F$. So, $C \subset F$. Thus $F \in \mathscr{F}$. Hence $\mathscr{F}$ is a filter. Thus $\mathscr{C}$ is a filter base for the filter $\mathscr{F}$.

Conversely, suppose $\mathscr{C}$ is a filter base for a filter $\mathscr{F}$. Clearly, $\mathscr{C} \subset \mathscr{F}$. Hence for $C, D \in \mathscr{C}$, $C, D \in \mathscr{F} \Rightarrow C \cap D \in \mathscr{F} \Rightarrow$ there is $B \in \mathscr{C}$ such that $B \subset C \cap D$. This completes the proof. □

35.6. Definition. A filter $\mathscr{F}$ on a set X is said to be a *free filter* if $\cap\mathscr{F} = \emptyset$. A filter which is not a free filter is called a *fixed filter*.

35.7. Definition. A filter $\mathscr{F}$ is called a *maximal filter* or an *ultrafilter* if for any filter $\mathscr{G}$, $\mathscr{F} \leq \mathscr{G} \Rightarrow \mathscr{F} = \mathscr{G}$.

35.8. Theorem. *Let X be a set. A filter $\mathscr{F}$ on X is an ultrafilter if and only if for each $A \subset X$, either $A \in \mathscr{F}$ or $X \setminus A \in \mathscr{F}$.*

Proof. Suppose $\mathscr{F}$ is an ultrafilter and $A \subset X$. Suppose $A \notin \mathscr{F}$. Let $E \in \mathscr{F}$. If $E \subset A$, then $A \in \mathscr{F}$. So, $E \not\subset A$. But then $E \cap (X \setminus A) \neq \emptyset$. Let

$$\mathscr{C} = \{(X \setminus A) \cap E : E \in \mathscr{F}\}. \tag{35.8.1}$$

Let $(X \setminus A) \cap E, (X \setminus A) \cap F \in \mathscr{C}$, with $E, F \in \mathscr{F}$. Then $E \cap F \in \mathscr{F}$. Thus $(X \setminus A) \cap (E \cap F) \in \mathscr{C}$ such that $(X \setminus A) \cap (E \cap F) \subset (X \setminus A) \cap E \cap (X \setminus A) \cap F$. Thus $\mathscr{C}$ is a filter base. Let $\mathscr{G}$ be the filter generated by $\mathscr{C}$. Then for $E \in \mathscr{F}$, $(X \setminus A) \cap E \in \mathscr{C}$ and $(X \setminus A) \cap E \subset E$. Hence $E \in \mathscr{G}$. Thus $\mathscr{F} = \mathscr{G}$. But then $(X \setminus A) = (X \setminus A) \cap X \in \mathscr{C} \subset \mathscr{G}$ so that $(X \setminus A) \in \mathscr{G} = \mathscr{F}$. This proves that either A or $X \setminus A$ is in $\mathscr{F}$.

Conversely, suppose $\mathscr{F}$ is not an ultrafilter. Let $\mathscr{G}$ be a filter such that $\mathscr{F} \subsetneq \mathscr{G}$. Let $A \in \mathscr{G}$ be such that $A \notin \mathscr{F}$. If $X \setminus A \in \mathscr{F}$, then $X \setminus A, A \in \mathscr{G} \Rightarrow \emptyset \in \mathscr{G}$, a contradiction. Thus neither A nor $X \setminus A$ is in $\mathscr{F}$. This completes the proof. □

35.9. Theorem. *Let $\mathscr{F}$ be an ultrafilter on a set X. Then $\mathscr{T} = \mathscr{F} \cup \{\emptyset\}$ is a door space.*

Proof. By the Theorem 35.2, $\mathscr{T}$ is a topological space. Now let A be a proper subset of X. Since $\mathscr{F}$ is an ultrafilter, $A \in \mathscr{F}$ or $X \setminus A \in \mathscr{F}$. Equivalently, A is $\mathscr{T}$-open or A is $\mathscr{T}$-closed. □

35.10. Theorem. *Every filter is contained in an ultrafilter.*

PROOF. Let X be a set and $\mathscr{F}$ be a filter. Let $\mathscr{A} = \{\mathscr{G} : \mathscr{G}$ is a filter on X and $\mathscr{F} \leq \mathscr{G}\}$. One easily verifies that $\mathscr{A}$ is a poset. Let $\mathscr{C}$ be a chain in $\mathscr{A}$. Let $\mathscr{G} = \cup\mathscr{C}$. Clearly, $\mathscr{G}$ is a nonempty collection of nonempty subsets of X. For $E, F \in \mathscr{G}$, there are filters $\mathscr{F}_1, \mathscr{F}_2$ such that $E \in \mathscr{F}_1$, $F \in \mathscr{F}_2$. Let $\mathscr{F}_1 \leq \mathscr{F}_2$. But then $E, F \in \mathscr{F}_2$, and hence, $E \cap F \in \mathscr{F}_2 \subset \mathscr{G}$. Also for $E \in \mathscr{G}$ and $E \subset F$, we have $E \in \mathscr{F}_1$ for some filter $\mathscr{F}_1 \in \mathscr{C}$. But then $F \in \mathscr{F}_1 \subset \mathscr{G}$. Thus $\mathscr{G}$ is an upper bound of $\mathscr{C}$. Thus every chain in $\mathscr{A}$ has an upper bound in $\mathscr{A}$. Hence, by Zorn's Lemma, $\mathscr{A}$ has a maximal element, say $\mathscr{G}$. Suppose now that $\mathscr{G} \leq \mathscr{G}_1$ for some filter $\mathscr{G}_1$. But then $\mathscr{F} \leq \mathscr{G}_1$, and hence, $\mathscr{G}_1 \in \mathscr{A}$. Since $\mathscr{G}$ is a maximal element of $\mathscr{A}$, $\mathscr{G}$ is a maximal filter. Thus $\mathscr{F}$ is contained in a maximal filter. □

35.11. Definition. Let X be a topological space and $x \in X$. The filter $\mathscr{U}_x$ on X is called the *neighbourhood filter at* x.

35.12. Example.

(1) Let X be a nonempty set and $A \subset X$. Then $\mathscr{C} = \{A\}$ is a filter base and it generates the filter $\mathscr{F} = \{F \subset X : A \subset F\}$.
(2) As a particular case of the above, for an $x \in X$, the filter $\mathscr{D}_x = \{F \subset X : x \in F\}$ is the neighbourhood filter at x for the discrete topology.
(3) For a set X, $\{X\}$ is a filter called the *indiscrete neighbourhood filter*.
(4) $\mathscr{C} = \{(a, \infty) : a \in \mathbb{R}\}$ is a filter base and the filter generated by $\mathscr{C}$ is called the *Fréchet filter*.
(5) $\mathscr{C} = \{(0, \frac{1}{n}) : n \in \mathbb{N}\}$ is a filter base.

35.13. Definition. Let X be a topological space, $x \in X$ and $\mathscr{F}$ be a filter on X. We say that $\mathscr{F}$ *converges to* x and write $\mathscr{F} \to x$ if $\mathscr{U}_x \leq \mathscr{F}$.

35.14. Definition. Let X be a topological space, $x \in X$ and $\mathscr{F}$ be a filter on X. We say that $\mathscr{F}$ *clusters at* x if $U \in \mathscr{U}_x, F \in \mathscr{F} \Rightarrow U \cap F \neq \emptyset$.

35.15. Remark. (1) $\mathscr{U}_x \to x$. Also, (2) $\mathscr{F} \to x, \mathscr{F} \leq \mathscr{G} \Rightarrow \mathscr{G} \to x$.

35.16. Theorem. *Let X be a topological space, $x \in X$ and $\mathscr{F}$ be a filter on X. The following are equivalent.*

(1) $\mathscr{F}$ clusters at x.
(2) There is a finer filter than $\mathscr{F}$ which converges to x.

In particular, if $\mathscr{F} \to x$, then $\mathscr{F}$ clusters at x.

PROOF. Suppose $\mathscr{F}$ clusters at x. Define $\mathscr{C} = \{F \cap U : F \in \mathscr{F}, U \in \mathscr{U}_x\}$. Then $F \cap U \neq \emptyset$ for every $F \in \mathscr{F}$ and $U \in \mathscr{U}_x$. Also, $X \in \mathscr{C}$. Thus $\mathscr{C}$ is a nonempty collection of nonempty sets. Since $\mathscr{F}$ and $\mathscr{U}_x$ are closed under finite intersections,

so is $\mathscr{C}$, and hence, $\mathscr{C}$ is a filter base for some filter, say $\mathscr{G}$. Also, for $E \in \mathscr{F}$, $E = E \cap X \in \mathscr{C} \subset \mathscr{G}$ gives $\mathscr{F} \leq \mathscr{G}$. On the other hand, for every $U \in \mathscr{U}_x$, $U = U \cap X \in \mathscr{C} \subset \mathscr{G}$ gives $\mathscr{G} \to x$.

Conversely, suppose there is a filter $\mathscr{G} \geq \mathscr{F}$ such that $\mathscr{G} \to x$. But then $\mathscr{F} \subset \mathscr{G}$ and $\mathscr{U}_x \subset \mathscr{G}$ guarantees that $F \in \mathscr{F}$, $U \in \mathscr{U}_x \Rightarrow F \cap U \neq \emptyset$. Thus $\mathscr{F}$ clusters at x. □

35.17. Theorem. *An ultrafilter $\mathscr{F}$ clusters at x if and only if $\mathscr{F} \to x$.*

Proof. Let an ultrafilter $\mathscr{F}$ cluster at x and $U \in \mathscr{U}_x$. Either U or $X \setminus U$ is in $\mathscr{F}$. Suppose $X \setminus U \in \mathscr{F}$. Since $\mathscr{F}$ clusters at x, U intersects with each member of $\mathscr{F}$, and hence, with $X \setminus U$, a contradiction. Thus $U \in \mathscr{F}$. That is, $\mathscr{U}_x \leq \mathscr{F}$. So, $\mathscr{U}_x \to x$. □

35.18. Theorem. *Let X be a topological space, $A \subset X$ and $x \in X$. The following are equivalent.*

(1) $x \in \overline{A}$.
(2) There is a filter $\mathscr{F}$ such that $A \in \mathscr{F}$ and $\mathscr{F} \to x$.

Proof. Suppose $x \in \overline{A}$. Define $\mathscr{C} = \{A \cap U : U \in \mathscr{U}_x\}$. It is easy to verify that $\mathscr{C}$ is a filter base for some filter on X, say $\mathscr{F}$. Then $A \in \mathscr{F}$ and $\mathscr{F} \to x$.

Conversely, suppose there is a filter $\mathscr{F}$ on X such that $\mathscr{F} \to x$ and $A \in \mathscr{F}$. But then $U \in \mathscr{U}_x \Rightarrow U \in \mathscr{F} \Rightarrow U \cap A \neq \emptyset$. Thus $x \in \overline{A}$. □

35.19. Proposition. *Let X, Y be sets, $f : X \to Y$ be a function and $\mathscr{F}$ be a filter on X. Then*

$$\mathscr{C} = \{f(E) : E \in \mathscr{F}\}. \tag{35.19.1}$$

is a filter base on Y.

Proof. Let $f(E), f(F) \in \mathscr{C}$ with $E, F \in \mathscr{F}$. Then $E \cap F \in \mathscr{F} \Rightarrow f(E \cap F) \in \mathscr{C}$. Also $f(E \cap F) \subset f(E) \cap f(F)$. This completes the proof. □

IV. N°$_t$ations: ☞ The filter generated by $\mathscr{C}$ given in (35.19.1) is denoted by $f(\mathscr{F})$.

35.20. Theorem. *Let X, Y be topological spaces, $f : X \to Y$ and $x \in X$. Then f is continuous at x if and only if $\mathscr{F} \to x \Rightarrow f(\mathscr{F}) \to f(x)$.*

Proof. Let f be continuous at x and $V \in \mathscr{U}_{f(x)}$. So, $f^{-1}(V) \in \mathscr{U}_x$. If $\mathscr{F} \to x$, $f^{-1}(V) \in \mathscr{F}$, giving $f^{-1}(f(V)) \in f(\mathscr{F})$. So, $V \in f(\mathscr{F})$. That is, $f(\mathscr{F}) \to f(x)$.

Conversely, let $V \in \mathscr{U}_{f(x)}$. Since $\mathscr{U}_x \to x$, $f(\mathscr{U}_x) \to x$. So, $V \in f(\mathscr{U}_x)$. Hence there is $U \in \mathscr{U}_x$ such that $f(U) \subset V$. Thus f is continuous at x. □

35.21. Theorem. *Image of an ultrafilter is an ultrafilter.*

PROOF. Let X, Y be sets, $f\colon X \to Y$ be a function and $\mathscr{F}$ be an ultrafilter on X. Let $E \subset Y$. Then $f^{-1}(E) \in \mathscr{F} \Rightarrow f(f^{-1}(E)) \in f(\mathscr{F})$. Since $f(f^{-1}(E)) \subset E$, $E \in f(\mathscr{F})$. Similarly, if $f^{-1}(X \setminus E) \in \mathscr{F}$, then $X \setminus E \in f(\mathscr{F})$. Thus $f(\mathscr{F})$ is an ultrafilter. □

35.22. Theorem. *Let $\{(X_i, \mathscr{T}_i) : i \in I\}$ be a family of topological spaces and $X = \prod_{i\in I} X_i$ be the product space and $\mathbf{x} \in X$. A filter $\mathscr{F}$ in X converges to $\mathbf{x}$ if and only if $\pi_i(\mathscr{F}) \to \pi_i(\mathbf{x})$ for each $i \in I$.*

PROOF. One way proof follows from the Theorem 35.20. Now suppose $\pi_i(\mathscr{F}) \to \pi_i(\mathbf{x})$ for each $i \in I$. Let $S = \pi_j^{-1}(V) \in \mathscr{U}_{\mathbf{x}}$ with $V \in \mathscr{T}_j$. But then $V \in \mathscr{U}_{\mathbf{x}_i}$. So, $V \in \pi(\mathscr{F})$. Hence, there is $U \in \mathscr{F}$ such that $\pi_j(U) \subset V$. That is, $U \subset \pi_j^{-1}(V) = S$. Thus $S \in \mathscr{F}$. Let $B = S_1 \cap S_2 \cap S_3 \cap \cdots \cap S_n$ be a basic neighbourhood of $\mathbf{x}$ with $S_1, S_2, S_3, \ldots, S_n \in \mathscr{S}_\pi$. But then each $S_k \in \mathscr{F}$, and hence, $B \in \mathscr{F}$. Since every $U \in \mathscr{U}_x$ contains a basic neighbourhood of x, we conclude that $\mathscr{F} \to \mathbf{x}$. □

Problems

213. Let X be a set and $A \subset X$ be infinite. Show that

$$\mathscr{C} = \{E \subset A : A \setminus E \text{ is finite }\}$$

is a filter base for a free filter on X.

214. Let $\mathscr{F}$ be a filter on a set X. Show that $\mathscr{F}$ is an ultrafilter if and only if $E \cup F \in \mathscr{F} \Rightarrow E \in \mathscr{F}$ or $F \in \mathscr{F}$.

215. Let X be a nonempty set and $\mathscr{F}$ be a fixed filter. Show that $\mathscr{F}$ is an ultrafilter if and only if there is some $x \in X$ such that $\mathscr{F} = \{E \subset X : x \in E\}$.

216. Let $(X, \mathscr{T}_f)$ be a cofinite topological space and X be infinite. Let $\mathscr{F}_f = \mathscr{T}_f \setminus \{\emptyset\}$. Show that $\mathscr{F}_f$ is a filter that converges to every point in $(X, \mathscr{T}_f)$. Does $\mathscr{F}_f$ converge in X with the cocountable topology?

217. Show that a set X is finite if and only if all filter on X are fixed.

218. Let X be a nonempty set and $\mathscr{F}_c = \mathscr{T}_c \setminus \{\emptyset\}$. Show that $\mathscr{F}_c$ is a filter that converges to every point in $(X, \mathscr{T}_c)$. Does $\mathscr{F}_c$ converge in X with the discrete topology?

219. Let $\emptyset \neq A \subset \mathbb{R}$ and $\mathscr{F} = \{E \subset \mathbb{R} : A \subset E\}$. Show that $\mathscr{F}$ clusters at x if and only if $x \in \overline{A}$.

220. Let X be a set and $\{\mathscr{F}_i : i \in I\}$ be a family of filters on X. Show that $\cap_{i\in I}\mathscr{F}_i$ is a filter.

221. Let X be a topological space, $x \in X$ and $A \subset X$. Show that $\mathscr{C} = \{A \cap U : U \in \mathscr{U}_x\}$ is a filter if and only if $x \in A^\circ$ if and only if $\mathscr{C} = \mathscr{U}_x$. Also, show that $\mathscr{C}$ is a filter base if and only if $x \in \overline{A}$.

222. Let X, Y be sets, $f: X \to Y$ be a function. Show that

$$\{f^{-1}(E) : E \in \mathscr{F}\}.$$

is a filter base if and only if $E \cap f(X) \neq \emptyset$ for every $E \in \mathscr{F}$.

223. Let X be sets and $A \subset X$ and $\mathscr{F}$ be a filter on X. Define

$$\mathscr{F}_A = \{E \cap A : E \in \mathscr{F}\}.$$

Show that $\mathscr{F}_A$ is a filter on A if and only if $E \cap A \neq \emptyset$ for every $E \in \mathscr{F}$ if and only if there is a filter $\mathscr{G}$ on X such that $\mathscr{F} \leq \mathscr{G}$ and $A \subset \mathscr{G}$.

36. Relation between Nets and Filters

36.1. Theorem. *Let X be a nonempty set and $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in X. Then*

$$\mathscr{C} = \{T_\mu(\{x_\lambda\}) : \mu \in \Lambda\} \tag{36.1.1}$$

is a filter base.

Proof. Of course, tail of a net is nonempty. For $\lambda, \mu \in \Lambda$, there is $\gamma \in \Lambda$ such that $\gamma \geq \mu, \nu$. So, $T_\gamma(\{x_\lambda\}) \subset T_\mu(\{x_\lambda\}) \cap T_\nu(\{x_\lambda\})$. Thus $\mathscr{C}$ is a filter base. □

36.2. Definition. For a net $\{x_\lambda\}$ in a set X, the filter generated in (36.1.1) is called the *filter generated by* $\{x_\lambda\}$ and is denoted by $\mathscr{F}(\{x_\lambda\})$.

36.3. Theorem. *Let X be a topological space and $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in X. For $x \in X$, $x_\lambda \to x$ if and only if $\mathscr{F}(\{x_\lambda\}) \to x$.*

Proof. $x_\lambda \to x$ if and only if every $U \in \mathscr{U}_x$ contains some $T_\mu(\{x\lambda\})$ if and only if $U \in \mathscr{F}(\{x_\lambda\})$ if and only if $\mathscr{F}(\{x_\lambda\}) \to x$. □

36.4. Theorem. *Let X be a topological space and $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in X. For $x \in X$, $\{x_\lambda\}$ clusters at x if and only if $\mathscr{F}(\{x_\lambda\})$ clusters at x.*

Proof. $\{x_\lambda\}$ clusters at x if and only if every $U \in \mathscr{U}_x$ intersects every $T_\mu(\{x_\lambda\})$ if and only if every $U \in \mathscr{U}_x$ intersects with every $E \in \mathscr{F}(\{x_\lambda\})$ if and only if $\mathscr{F}(\{x_\lambda\})$ clusters at x. □

36.5. Theorem. *Let X be a topological space and $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in X. Let $\{x_{\varphi(\gamma)}\}_{\gamma\in\Gamma}$ be its subnet. Then $\mathscr{F}(\{x_\lambda\}) \leq \mathscr{F}(\{x_{\varphi(\gamma)}\})$.*

Proof. Let $E \in \mathscr{F}(\{x_\lambda\})$. So, $T_\mu(\{x_\lambda\}) \subset E$ for some $\mu \in \Lambda$. Taking $\delta \in \Gamma$ with $\varphi(\delta) \geq \mu$, we see that $T_\delta(\{x_{\varphi(\gamma)}\}) \subset T_\mu(\{x_\lambda\}) \subset E$. Thus $E \in \mathscr{F}(\{x_{\varphi(\gamma)}\})$, giving $\mathscr{F}(\{x_\lambda\}) \leq \mathscr{F}(\{x_{\varphi(\gamma)}\})$. □

36.6. Theorem. *Let X be a topological space. A net $\{x_\lambda\}_{\lambda\in\Lambda}$ in X is an ultranet if and only if $\mathscr{F}(\{x_\lambda\})$ is an ultrafilter.*

PROOF. $\{x_\lambda\}$ is an ultranet eventually in $E \subset X$ if and only if $T_\mu(\{x_\lambda\}) \subset E$ for some μ if and only if $E \in \mathscr{F}(\{x_\lambda\})$. The same implication holds for $X \setminus E$. Thus the result follows. □

The following is easy to see.

36.7. Lemma. *Let X be a set and $\mathscr{F}$ be a filter on X. Define*

$$\Lambda_{\mathscr{F}} = \{(a, F) : a \in F \in \mathscr{F}\}. \tag{36.7.1}$$

For $(a, F), (b, E) \in \Lambda_{\mathscr{F}}$, define $(a, F) \le (b, E)$ if $E \subset F$. Then $\Lambda_{\mathscr{F}}$ is directed.

36.8. Definition. For a filter $\mathscr{F}$ on a set Xj and $\Lambda_{\mathscr{F}}$ as in (36.7.1), the net $(a, F) \in \Lambda_{\mathscr{F}} \mapsto x_{(a,F)} = a \in X$ is called the *net based on the filter $\mathscr{F}$*.

36.9. Theorem. *Let X be a topological space and $\mathscr{F}$ be a filter on X and $x \in X$. Then $\mathscr{F} \to x$ if and only if the net $\{x_{(a,F)}\} \to x$.*

PROOF. If $\mathscr{F} \to x$, then $U \in \mathscr{U}_x, (y, V) \ge (x, U) \Rightarrow x_{(y,U)} = y \in V \subset U$, proving $x_{(a,F)} \to x$.

Conversely, suppose $x_{(a,F)} \to x$. Let $V \in \mathscr{U}_x$. Then there is $(a_0, U_0) \in \Lambda_{\mathscr{F}}$ such that $x_{(a,U)} \in V$ whenever $(a, U) \ge (a_0, U_0)$. In particular, $a = x_{(a,U_0)} \in V$ for all $a \in U_0$, giving $U_0 \subset V$. Since $U_0 \in \mathscr{F}$, the result follows. □

36.10. Theorem. *Let X be a topological space and $\mathscr{F}$ be a filter on X and $x \in X$. Then $\mathscr{F}$ clusters at x if and only if the net $\{x_{(a,F)}\}$ clusters at x.*

PROOF. Let $\mathscr{F}$ cluster at x. Let $U \in \mathscr{U}_x$ and $(a, F) \in \Lambda_{\mathscr{F}}$. Then $U \cap F \neq \emptyset$. Let $b \in U \cap F$. Then $(b, F) \ge (a, F)$ and $b = x_{(b,F)} \in U$ Thus $\{x_{(a,F)}\}$ clusters at x.

Conversely, suppose $\{x_{(a,F)}\}$ clusters at x. Let $U \in \mathscr{U}_x$ and $E \in \mathscr{F}$. Let $b \in E$. Then there is $(a, F) \ge (b, E)$ such that $a = x_{(a,F)} \in U$. So, $F \subset E$ and $a \in F \cap U$. Thus $a \in U \cap E$. Hence $\mathscr{F}$ clusters at x. This completes the proof. □

36.11. Theorem. *Let X be a topological space and $\mathscr{F}$ be a filter on X and $x \in X$. Then $\mathscr{F}$ is ultrafilter if and only if the net $\{x_{(a,F)}\}$ is an ultrafilter.*

PROOF. It follows from the fact that $E \in \mathscr{F}$ if and only if $T_{(b,E)}\{x_{(a,F)}\} \subset E$. That is, $E \in \mathscr{F}$ if and only if $x_{(a,F)}$ is eventually in E. □

Problems

224. Let $\mathscr{F}$ be a fixed filter on a set X. Show that the net $\{x_{(a,F)}\}$ net based on $\mathscr{F}$ is an ultranet if and only if $\{x\} \in \mathscr{F}$. Show that we cannot weaken the condition $\{x\} \in \mathscr{F}$ by $x \in \cap\mathscr{F}$.

CHAPTER 7

Separation

The earliest definition of topology by Hausdorff included a condition that made a all the topological spaces what we now know as a T_2-space or Hausdorff space. Later on the separation axioms were studied under the name of an *accessible space*, which we know now as a T_1-space. A T_1-space was baptised as a *Fréchet space* or a space with a Fréchet topology. Later on mathematicians settled with the name *T_1-space*. Gradually the series of names *T_1-space*, *T_2-space*, *T_3-space* and *T_4-space* was used for spaces with different separation properties. Actually the letter T was used by Tietze in 1923. Of course T has nothing to do with his name, but it is taken from the German word *Trennung* meaning separation. Besides these the terms a *regular space* and a *normal space* were also defined. Some books swapped a T_3-space with a regular space and a T_4-space with a normal space. Later on the series was appended with *T_0-space*, *$T_{2\frac{1}{2}}$-space*, *$T_{3\frac{1}{2}}$-space*, *T_5-space* and *T_6-space*. $T_{2\frac{1}{2}}$-space is also known as *completely Hausdorff space* and $T_{3\frac{1}{2}}$-space is also known as *completely regular space*. A T_0-space is also known as *Kolmogorov space*.

37. T_0-Spaces

37.1. Definition. A topological space $(X, \mathscr{T})$ is said to be a *T_0-space* if for every pair of distinct points $x, y \in X$, one of them has a neighbourhood not containing the other. That is, there is an open set U such that either $x \in U, y \notin U$ or $y \in U, x \notin U$.

37.2. Theorem. *A topological space X is T_0 if and only if for every distinct $x, y \in X$, $\overline{x} \neq \overline{y}$.*

Proof. Suppose X is T_0. Let $x, y \in X$ be two distinct points. Without loss of generality,let U be an open set containing x but not y. But then $x \in \overline{\{x\}} \setminus \overline{\{y\}}$.

Conversely, given two distinct points $x, y \in X$, assume that $z \in \overline{\{x\}} \setminus \overline{\{y\}}$. Since $z \notin \overline{\{y\}}$, there is an open set U such that $z \in U$ and $U \cap \{y\} = \emptyset$. However, since $z \in \overline{\{x\}}$, $x \in U$. This completes the proof.

□

37.3. Example.

(1) A metric space is a T_0-space.
(2) The Sierpiński space is a T_0-space.
(3) The usual topology, the lower limit topology, the upper limit topology are T_0-topologies.
(4) A set X with any of the left order topology, the right order topology, the cofinite topology, the cocountable topology, co α-topology, cosub α-topology, discrete topology, excluded point topology, particular point topology is a T_0-space.
(5) Indiscrete space with at least two points is not a T_0-space.
(6) A topology finer than a T_0-topology is also T_0.
(7) Intersection of two T_0-topologies on a set need not be T_0. Indeed the intersection of the two Sierspińsky topologies on $\{a, b\}$ is the indiscrete topology, and hence, is not T_0. Also the intersection of left order and the right order topologies on $\mathbb{Z}$ is the indiscrete topology.

37.4. Theorem. *Subspace of a T_0-space is a T_0-space.*

Proof. Let X be a T_0-space and $Y \subset X$. Let $x, y \in Y$ be distinct. Without loss of generality, we assume that there is a neighbourhood U of x in X that does not contain y. But then $U \cap X$ is a neighbourhood of x in Y which does not contain y. Thus Y is a T_0-space. □

37.5. Theorem. *Homeomorphic image of a T_0-space is a T_0-space.*

Proof. Let X be a T_0-space, Y be a topological space and $f\colon X \to Y$ be a homeomorphism. Let $x, y \in Y$ be two distinct points. Then $f^{-1}(x), f^{-1}(y)$ are two distinct points of X. Since X is T_0, without loss of generality, let U be a neighbourhood of $f^{-1}(x)$ that does not contain $f^{-1}(y)$. But then $f(U)$ is a neighbourhood of x that does not contain y. Thus Y is T_0. □

37.6. Theorem. *Let $X = \prod_{i\in I} X_i$ be a product space. Then X is T_0 if and only if each X_i is T_0.*

Proof. One way is obvious as X_i is homeomorphic to a subspace of X, and subspace and homeomorphic image of a T_0-space are T_0-spaces.

Conversely, suppose each X_i is T_0. Let $\mathbf{x}, \mathbf{y} \in X$ be distinct. Let $j \in I$ be such that $\mathbf{x}_j \neq \mathbf{y}_j$. Since X_j is T_0, we take, without loss of generality, a neighbourhood

U of $\mathbf{x}_j$ that does not contain $\mathbf{x}_j$. But then $\pi_j^{-1}(U)$ is a neighbourhood of $\mathbf{x}$ that does not contain $\mathbf{y}$. This completes the proof. □

37.7. Example. A quotient of a T_0-space need not be T_0.

Indeed consider the characteristic function $\chi_{\mathbb{Q}}\colon \mathbb{R} \to \{0,1\}$, where $\mathbb{R}$ carries the usual topology. Then the quotient topology on $\{0,1\}$ is the indiscrete topology, which is not T_0.

Open or closed quotient of a T_0-space is not T_0 as the following example shows.

37.8. Example. Let $\mathscr{T}$ be the left order topology on $\mathbb{Z}$. Then $\mathbb{Z}$ is T_0. Indeed for $m < n \in \mathbb{Z}$ $(-\infty, n)$ is an open set containing m but not n. Define $f\colon \mathbb{Z} \to \{0,1\}$ by $f(n) = n \mod 2$. Then the quotient topology $\mathscr{T}_q(f)$ is indiscrete. Since every nonempty open or closed set contains even as well as odd integers, image of every nonempty open or closed set is $\{0,1\}$. Thus f is a closed as well as open map. But $\{0,1\}$ is not T_0.

Problems

225. Let X be a topological space. Define a relation $\sim$ on X by $x \sim y$ if open sets containing x are the same as the open sets containing y. Show that

(1) $\sim$ is an equivalence relation.
(2) $X/\sim$ is T_0.
(3) If X is T_0, then $X \cong X/\sim$.

226. Let X be a set, Y be a discrete space and $f\colon X \to Y$ be a function. Show that X with the weak topology given by f is T_0 if and only if it is discrete.

38. T_1-Spaces

38.1. Definition. A topological space $(X, \mathscr{T})$ is said to be a *T_1-space* if for every pair of distinct points $x, y \in X$, each of them has a neighbourhood not containing the other. That is, there are two open sets U, V in X such that $x \in U$, $y \in V$, $x \notin V$, $y \notin U$.

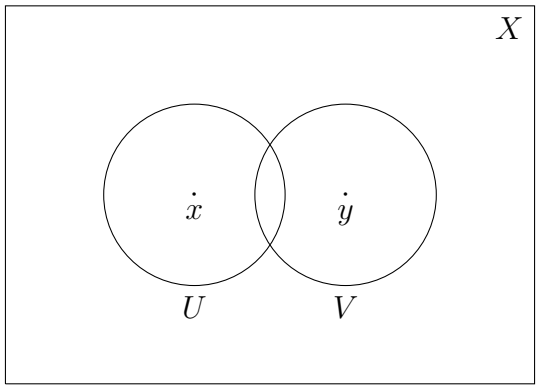


FIGURE 28. Schematic T_1-space

38.2. Example. From the definitions of a T_0-space and a T_1-space, it is clear that every T_1-space is a T_0-space. However, Sierspińsky space is T_0 but not T_1.

38.3. Theorem. *A topological space X is a T_1-space if and only if every singleton in X is closed in X.*

PROOF. Let X be a T_1-space. Let $x, y \in X$ be distinct and U, V be open sets such that $x \in U$, $y \in V$, $x \notin V$ and $y \notin U$. Then V is an open set containing y and not intersecting with $\{x\}$. Thus y is not a limit point of $\{x\}$. Consequently, no point of X other than x can be a limit point of $\{x\}$. Thus $\{x\}$ is closed.

Suppose now that every singleton is closed in X. Let $x, y \in X$ be distinct. Let $U = X \setminus \{y\}$ and $V = X \setminus \{x\}$. Then U, V are open subsets of X such that $x \in U$, $y \in V$, $x \notin V$, $y \notin U$. Thus X is T_1. □

38.4. Corollary. *An ordered space is T_1.*

PROOF. Let X be an ordered space and $x \in X$. Then $X \setminus \{x\} = (-\infty, x) \cup (x, \infty)$, an open set. Hence $\{x\}$ is closed. Thus every singleton in X is closed, and hence, X is T_1. □

38.5. Corollary. *A topological space X is a T_1-space if and only if every finite subset of X is closed in X.*

38.6. Corollary. *A metric space is T_1.*

PROOF. Let (X, d) be a metric space and $x \in X$. For any $y \neq x$, $r = d(x, y) > 0$. Hence $B(y, r) \cap \{x\} = \emptyset$. Thus $y \notin \overline{\{x\}}$; so that $\{x\}$ is closed. This completes the proof. □

38.7. Corollary. *A topological space $(X, \mathscr{T})$ is T_1 if and only if $\mathscr{T}_f \leq \mathscr{T}$.*

PROOF. Assume $\mathscr{T}_f \leq \mathscr{T}$. Since every singleton is $\mathscr{T}_f$-closed, so is the case in $\mathscr{T}$ too. Thus $\mathscr{T}$ is a T_1-topology.

Conversely, suppose $(X, \mathscr{T})$ is T_1. Let $G \subset X$ be $\mathscr{T}_f$-open. So $G = \emptyset$ or $X \setminus G$ is finite. If $G = \emptyset$, then $G \in \mathscr{T}$. If $X \setminus G$ is finite, then $X \setminus G$ is $\mathscr{T}$-closed. That is, $G \in \mathscr{T}$. Thus $\mathscr{T}_f \leq \mathscr{T}$. □

38.8. Corollary. *Let X be a T[1]space, $A \subset X$ and $x \in A'$. Then every neighbourhood U of x contains infinitely many points of A.*

PROOF. Suppose $U \in \mathscr{U}_x$ and $U \cap A \setminus \{x\} = \{x_1, x_2, x_3, \ldots, x_n\}$. But then $U \cap (\cap_{k=1}^{n} (X \setminus \{x_k\}))$ is a neighbourhood of x that does not contain any point of A, a contradiction. Thus $U \cap A$ must be infinite. □

38.9. Corollary. *Let X be a T_1-space and $A \subset X$. Then A' is closed.*

PROOF. Let $x \in A''$ and U be a neighbourhood of x. Then U contains infinitely many points of A', say $y \in U \cap A' \setminus \{x\}$. Then U is a neighbourhood of y and $y \in A'$. So, U contains infinitely many points of A. Consequently, U contains a point of A other than x. Thus $x \in A'$. That is, $A'' \subset A'$, and hence, A' is closed. □

38.10. Proposition. *In a topological space X if convergent sequences have unique limits, then X is T_1. However the converse does not hold.*

PROOF. Let X be a topological space. Suppose X is not T_1. So, there is a singleton, say $\{x\}$, in X such that $\{x\}$ is not closed. Let $y \in \overline{\{x\}}$. Then the constant sequence $\{x, x, x, \dots\}$ converges to x as well as y. On the other hand, $\mathbb{R}$ with the cofinite topology is T_1. But the sequence $\{n\}$ converges to each and every point of $\mathbb{R}$. □

38.11. Proposition. *A finite T_1-space is discrete.*

PROOF. Let $(X, \mathscr{T})$ be a finite T_1-space. Hence by the Example 7.11, $\mathscr{D} \leq \mathscr{T}_f \leq \mathscr{T}$. Hence $\mathscr{T} = \mathscr{D}$. □

In fact, the following result of the similar color is more interesting.

38.12. Theorem. *A topological space X is T_1 if and only if every finite subset of X is discrete.*

PROOF. Suppose X is T_1 and $F \subset X$ is finite. Then F is a finite T_1-space, and hence, is discrete.

Conversely, assume that every finite subset of X is discrete. Let $x, y \in X$ be distinct. Then, $\{x, y\}$ is discrete. So, $\{x\}$ and $\{y\}$ are open sets in $\{x, y\}$. Hence there exist open sets U, V in X such that $U \cap \{x, y\} = \{x\}$, that is $x \in U$ and $y \notin U$. Also $V \cap \{x, y\} = \{y\}$, that is $y \in V$ and $x \notin V$. Thus X is T_1. □

38.13. Theorem. *A topological space X is T_1 if and only if every subset of X is an intersection of open sets.*

PROOF. Suppose X is T_1. Let $G \subset X$. Then $G = \cap_{x \notin G}(X \setminus \{x\})$ is an intersection of open sets.

Conversely, suppose every subset of X is an intersection of open sets. Let $x, y \in X$. Clearly, $\{y\}$ is intersection of open sets containing y. So there is an open set containing y not containing x. So, y is not a limit point of $\{x\}$. Thus $\{x\}$ is closed. Thus X is T_1. □

38.14. Theorem. *Subspace of a T_1-space is T_1.*

PROOF. Let X be a T_1-space and $Y \subset X$. Let $x \in Y$. Then $x \in X$. Since X is T_1, $\{x\}$ is closed in X. But then $\{x\} = \{x\} \cap Y$ is closed in Y. Thus every singleton is closed in Y. So, Y is T_1. This completes the proof. □

38.15. Theorem. *Homeomorphic image of a T_1-space is a T_1-space.*

PROOF. Let X be a T_1-space and $f\colon X \to Y$ be a homeomorphism. Let $y \in Y$. The $\{f^{-1}(y)\}$ is a singleton in X and since X is T_1, $\{f^{-1}(y)\}$ is closed in X. But a homeomorphism is a closed map so that $\{y\}$ is closed. Thus Y is T_1. □

38.16. Theorem. *Let $X = \prod_{i\in I} X_i$ be a product space. Then X is T_1 if and only if each X_i is T_1.*

PROOF. X_i, being homeomorphic to a subspace of X, is T_1, when X is.

Note that for $\mathbf{x} \in X$, $\{\mathbf{x}\} = \cap_{i\in I}\pi_i^{-1}(\{\mathbf{x}_i\})$. If each X_i is T_1, then each $\{\mathbf{x}_i\}$ is closed in X_i, which ensures that $\{\mathbf{x}\}$ is closed in X, and hence, X is T_1. □

38.17. Example.

(1) A metric space is a T_1-space.
(2) The Sierpiński space is not a T_1-space.
(3) The usual topology, the lower limit topology, the upper limit topology are T_1-topologies.
(4) A set X with any of the left order topology, the right order topology, excluded point topology, particular point topology is not a T_1-space.
(5) A set X with any of the cofinite topology, the cocountable topology, co α-topology, cosub α-topology, discrete topology is a T_1-space.
(6) Indiscrete space with at least two points is not a T_1-space.
(7) A topology finer than a T_1-topology is also T_1.
(8) Intersection of any number of T_1-topologies on a set is T_1.
(9) The cofinite topology on a set X is the smallest T_1-topology on X.

38.18. Example. Quotient of a T_1-space need not be T_1. Indeed, consider the quotient of $\mathbb{R}$ as in the Example 37.7. The quotient topology of $\{0, 1\}$ given by $\chi_{\mathbb{Q}}$ is the indiscrete topology, which is not T_1.

38.19. Example. A closed quotient of a T_1-space is T_1. Indeed, let $f\colon X \to Y$ be a closed quotient map. Let $y \in Y$. Then there is $x \in X$ such that $f(x) = y$. Since X is T_1, $\{x\}$ is closed. But then $\{y\} = f(\{x\})$ is closed in Y. Thus every singleton in Y is closed. Thus Y is T_1.

Problems

227. Let X be a set, Y be a discrete space and $f\colon X \to Y$ be a function. Show that X with the weak topology given by f is T_1 if and only if it is discrete.

228. Let X be a topological space and $x \in X$. Show that X is T_1 if and only if $X \setminus \{x\}$ is an open T_1-subspace of X.

229. Determine which of the following topologies makes $\mathbb{R}$ a T_1-space.

(1) The usual topology,
(2) Lower limit topology,
(3) Cocountable topology,
(4) Cofinite topology and
(5) $\mathscr{T}_K$ defined in the Example 8.26.

39. T_2-Spaces

39.1. Definition. A topological space X is said to be a T_2-space if for every distinct points $x, y \in X$, there are two open sets U, V such that

$$x \in U, y \in V \text{ and } U \cap V = \emptyset.$$

A T_2-space is also known as *Hausdorff space*.

The definitions of T_1 and T_2-spaces reveal that every T_2-space is T_1, and hence, T_0.

39.2. Proposition. *A finite T_2-space is discrete.*

Proof. The Proposition 13.10 and the fact that every T_2-space is T_1 prove this. □

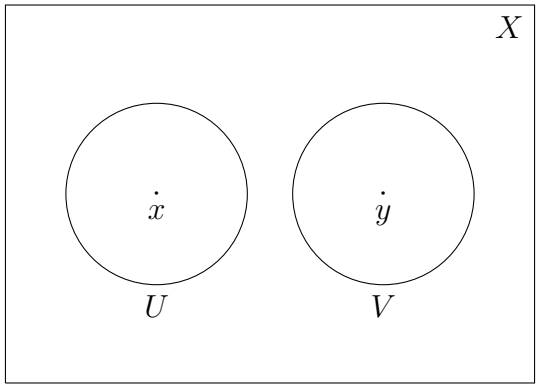


Figure 29. Schematic T_2-space

39.3. Example. A T_1-space need not be T_2. Indeed the cofinite space $(X, \mathscr{T}_f)$ is T_1 but it is T_2 if and only if X is finite. Thus $(\mathbb{R}, \mathscr{T}_f)$ is T_1 but it is not T_2.

39.4. Theorem. *A Hausdorff door space can have at most one limit point.*

Proof. Let X be a Hausdorff door space. Suppose $a, b \in X$ be two limit points of X. Let G, H be two disjoint open sets such that $a \in G, b \in H$. If possible,

suppose $U = (G \setminus a) \cup \{b\}$ is open. But then $U \cap H = \{b\}$, asserting that b is not a limit point. But this is not the case. Hence U must be closed. But then $X \setminus U$ is open. Hence $(X \setminus U) \cap G = \{a\}$ is open, showing that a is not a limit point of X. Thus X has at most one limit point. □

It should be noted that a discrete space is a door space without any limit point.

39.5. Theorem. *An ordered space is T_2.*

PROOF. Let X be an ordered space and $x < y$ be two elements of X. If there is $z \in X$ such that $x < z < y$, then $(-\infty, z)$ and (z, ∞) are two disjoint open subsets of X containing x and y respectively. If no such z exists, then x is an immediate predecessor of y. But then $(-\infty, y)$ and (x, ∞) are two disjoint open subsets of X containing x and y respectively. □

39.6. Theorem. *A metric space is T_2.*

PROOF. Let (X, d) be a metric space. Let $x, y \in X$ be distinct. So, $r = d(x, y) > 0$. Clearly, $B(x, \frac{r}{2})$ and $B(y, \frac{r}{2})$ are two open sets containing x and y respectively. Also $z \in B(x, \frac{r}{2}) \cap B(z, \frac{r}{2}) \Rightarrow d(x, y) \leq d(x, z) + d(z, y) < r$, a contradiction. Thus $B(x, \frac{r}{2})$ and $B(y, \frac{r}{2})$ are disjoint. This completes the proof. □

39.7. Theorem. *A topological space X is T_2 if and only if the diagonal Δ is closed in $X \times X$.*

PROOF. Suppose X is T_2. Let x, y be two distinct points of X and U, V be two open sets in X such that $x \in U, y \in V$ and $U \cap V = \emptyset$. Clearly, $U \times V$ is open in $X \times X$ and $(x, y) \in U \times V$.

$$\begin{aligned}(U \times V) \cap \Delta \neq \emptyset &\Leftrightarrow (z, z) \in U \times V \text{ for some } z \in X \\ &\Leftrightarrow z \in U, z \in V \\ &\Leftrightarrow U \cap V \neq \emptyset, \end{aligned} \tag{39.7.1}$$

a contradiction. Thus $(U \times V) \cap \Delta = \emptyset$.

Conversely, suppose Δ is closed in $X \times X$. Let x, y be two distinct points in X. So, $(x, y) \notin \Delta$. Hence, there are two open sets $U, V \subset X$ such that $U \times V \cap \Delta = \emptyset$. But then by (39.7.1), $U \cap V = \emptyset$. Thus X is T_2. □

39.8. Theorem. *A topological space X is T_2 if and only if every filter on X has at most one limit.*

PROOF. Suppose X is T_2 and $\mathscr{F}$ is a convergent filter on X. Suppose $\mathscr{F} \to x$ and $\mathscr{F} \to y$ for two distinct points $x, y \in X$. Let U, V be two open sets in X such that

$x \in U, y \in V$ and $U \cap V = \emptyset$. Since $\mathscr{F} \to x$, $U \in \mathscr{F}$, and since $\mathscr{F} \to y$, $V \in \mathscr{F}$. Since a filter is closed under finite intersection, $\emptyset = U \cap V \in \mathscr{F}$, a contradiction. Thus $\mathscr{F}$ cannot converge to more than one point.

Conversely, suppose every filter on X has at most one limit. Let $x, y \in X$ be distinct. Suppose $U \in \mathscr{U}_x, V \in \mathscr{U}_y \Rightarrow U \cap V \neq \emptyset$. Let

$$\mathscr{C} = \{U \cap V : U \in \mathscr{U}_x, V \in \mathscr{U}_y\}.$$

Let $U_1, U_2 \in \mathscr{U}_x, V_1, V_2 \in \mathscr{U}_y$. Then $U_1 \cap U_2 \in \mathscr{U}_x, V_1 \cap V_2 \in \mathscr{U}_y$. So, $U_1 \cap U_2 \cap V_1 \cap V_2 \in \mathscr{C}$. Thus $\mathscr{C}$ is a filter base, say for a filter $\mathscr{F}$. Clearly, $U \in \mathscr{U}_x \Rightarrow U = U \cap X \in \mathscr{C} \subset \mathscr{F}$. So, $\mathscr{U}_x \leq \mathscr{F}$. Similarly, $\mathscr{U}_y \leq \mathscr{F}$. So, $\mathscr{F} \to x, \mathscr{F} \to y$, a contradiction. Thus X is T_2. □

39.9. Theorem. *A topological space X is T_2 if and only if every net in X has at most one limit.*

Proof. Let X be a T_2-space and a net $\{x_\lambda\}$ in X converges to two distinct points $x, y \in X$. Let U, V be two disjoint open sets in X such that $x \in U, y \in V$. Since $x_\lambda \to x$, there is λ_0 such that $x_\lambda \in U$ for every $\lambda \geq \lambda_0$. Similarly since $x_\lambda \to y$, there is λ_1 such that $x_\lambda \in U$ for every $\lambda \geq \lambda_1$. Let λ be such that $\lambda \geq \lambda_1, \lambda \geq \lambda_2$. But then $x_\lambda \in U \cap V$, a contradiction. Thus $\{x_\lambda\}$ can have at most one limit.

Conversely, suppose every net in X has at most one limit. Suppose, if possible, that X is not T_2. So, $\Delta \subset X \times X$ is not closed in $X \times X$. Take a net $\{(x_\lambda, x_\lambda)\}$ in Δ such that $(x_\lambda, x_\lambda) \to (x, y) \notin \Delta$. So $x_\lambda \to x$ and $x_\lambda \to y$, a contradiction. Thus X must be T_2. □

39.10. Remark.

(1) We preferred to give the direct proof of the Theorem 39.9. Otherwise, one can argue that every net converges to at most one point if and only if every filter does so. Therefore, the result follows from the Theorem 39.8.
(2) Clearly, in a T_2-space every sequence can have at most one limit. However the converse fails in the cocountable topology.

Since sequences are also nets, we conclude that a sequence in a T_2-space has at most one limit. Also, by the Proposition 38.10, it is clear that the converse does not hold. However, in the presence of the first countability, the converse is true.

39.11. Proposition. *Let X be a first countable space. If every sequence in X has at most one limit, then X is T_2.*

Proof. Suppose X is not T_2. Let x, y be two points without disjoint neighbourhoods. That is each neighbourhood of x intersects each neighbourhood of y. Let $\{U_k : k \in \mathbb{N}\}$ and $\{V_k : k \in \mathbb{N}\}$ be local bases at x and y respectively. Note that for every $n \in \mathbb{N}$, $W_n = \cap_{k=1}^n (U_k \cap V_k) = (\cap_{k=1}^n U_k) \cap (\cap_{k=1}^n V_k) \neq \emptyset$. Also, $W_n \subset (\cap_{k=1}^n U_k) \subset U_k$ for every $1 \leq k \leq n$. Choose $x_n \in W_n$, $(n \in \mathbb{N})$. Let $W \in \mathscr{U}_x$. So, there is $m \in \mathbb{N}$ such that $U_m \subset W$. Now for $n \geq m$, $x_n \in W_n \subset (\cap_{k=1}^n U_k) \subset U_m \subset W$. Thus $x_n \to x$. In the same way, $x_n \to y$ also, a contradiction. Thus X must be T_2. □

39.12. Theorem. *Let X be a topological space and Y be a T_2-space. Let $f: X \to Y$ be a continuous function. Then f has a closed graph, that is $G(f)$ is closed in $X \times Y$.*

Direct proof. Let $(x, y) \in X \times Y$ but $(x, y) \notin G(f)$. That is, y and $f(x)$ are distinct points of Y. Since Y is T_2, there are two disjoint open subsets U, V of Y such that $y \in U$, $f(x) \in V$. By the continuity of f, we get an open set $G \subset X$ such that $f(G) \subset V$. Suppose $(x', y') \in G(f) \cap (G \times U)$. Hence $x' \in G$, $y' \in U$, $f(x') = y' \in V$. Thus $U \cap V \neq \emptyset$, a contradiction. Thus $G(f)$ must be closed. □

Proof Using nets. Let $(x, y) \in X \times Y$ be a limit point $G(f)$. So, there is a net $\{(x_\lambda, y_\lambda)\}$ from $G(f)$ such that $(x_\lambda, y_\lambda) \to (x, y)$. Hence $x_\lambda \to x$ and $y_\lambda \to y$. But since $(x_\lambda, y_\lambda) \in G(f)$, $y_\lambda = f(x_\lambda)$ for every λ. Hence, by the continuity of f, $f(x_\lambda) \to y$. By the uniqueness of the limits in T_2-space, $y = f(x)$. Thus $(x, y) \in G(F)$, proving the closedness of $G(F)$. □

39.13. Theorem. *Let X be a topological space and Y be a T_2-space. Let $f, g: X \to Y$ be continuous functions. Then $A = \{x \in X : f(x) = g(x)\}$ is closed.*

Proof. Consider $y \in X$ such that $f(y) \neq g(y)$. So, there are two disjoint open sets $U, V \subset Y$ such that $f(y) \in U$, $g(y) \in V$. But then $W = f^{-1}(U) \cap g^{-1}(V) \in \mathscr{U}_y$. Let $z \in W$ Then $f(z) \in U$ and $g(z) \in V$. Since $U \cap V = \emptyset$, $f(z) \neq g(z)$. Thus $W \cap A = \emptyset$. So, $y \notin \overline{A}$. Hence A is closed. □

39.14. Theorem. *A subspace of a T_2-space is T_2.*

Proof. Let X be a T_2-space and $Y \subset X$. Then $\Delta_X = \{(x, x) : x \in X\}$ is closed in $X \times X$. Hence $\Delta_Y = \Delta_X \cap (Y \times Y)$ is closed in $Y \times Y$. Thus Y is T_2. □

39.15. Theorem. *Homeomorphic image of a T_2-space is a T_2-space.*

Proof. Let X be a T_2-space and $f: X \to Y$ be a homeomorphism. Let $x, y \in Y$ be distinct. Then $f^{-1}(x), f^{-1}(y)$ are distinct in X. So that there exist two disjoint open sets $U, V \subset X$ such that $f^{-1}(x) \in U$ and $f^{-1}(y) \in V$. But a homeomorphism is an open map so that $f(U), f(V)$ are open in Y. Also, $x \in f(U), y \in f(V)$. Since f is one-one, $f(U) \cap f(V) = f(U \cap V) = \emptyset$. Thus Y is T_2. □

39.16. Theorem. *Let $X = \prod_{i \in I} X_i$ be a product space. Then X is T_2 if and only if each X_i is T_2.*

PROOF. X_i, being homeomorphic to a subspace of X, is T_2, when X is.

Conversely, assume that X_i is T_2 for every $i \in I$. Let $\mathbf{x}, \mathbf{y}$ be two distinct points of X. Choose $j \in I$ such that $\mathbf{x}_j \neq \mathbf{y}_j$. Let U, V be two disjoint open sets in X_j such that $\mathbf{x}_j \in U$, $\mathbf{x}_j \in V$. Then $\pi_j^{-1}(U), \pi_j^{-1}(V)$ are two open sets in X containing $\mathbf{x}, \mathbf{y}$ respectively. Also, $\mathbf{z} \in \pi_j^{-1}(U) \cap \pi_j^{-1}(V) \Rightarrow \mathbf{z}_j \in U \cap V$, a contradiction. Thus X is T_2. □

39.17. Example. A closed image of a T_2-space need not be T_2.

Indeed consider $\mathbb{R}_K = (\mathbb{R}, \mathscr{T}_K)$. Let $Y = \mathbb{R}_K/K$. Since K is $\mathscr{T}_K$-closed, the corresponding quotient map $f\colon \mathbb{R}_K \to Y$ is a closed map by the Theorem 21.26(2). Suppose there are two open sets U, V in Y such that $0 \in U$, $K \in V$ and $U \cap V = \emptyset$. But then $f^{-1}(U)$ and $f^{-1}(V)$ are two disjoint $\mathscr{T}_K$-open sets such that $0 \in f^{-1}(U)$ and $K \subset f^{-1}(V)$. Let B be a basic $\mathscr{T}_K$-open set such that $0 \in B \subset f^{-1}(U)$, If B is an interval, then for large enough k, $\frac{1}{k} \in B$, a contradiction. On the other hand, if $B = (a, b) \setminus K$, then choose large enough k such that $\frac{1}{k} < b$. Since $\frac{1}{k} \in f^{-1}(V)$, there is $(\frac{1}{k} - \varepsilon, \frac{1}{k} + \varepsilon) \subset f^{-1}(V)$. Taking ε small enough so that $\frac{1}{k+1} < \frac{1}{k} - \varepsilon$, we see that $\frac{1}{k} - \frac{\varepsilon}{2} \in f^{-1}(U) \cap f^{-1}(V)$, a contradiction. Hence $\mathbb{R}_K/K$ is not T_2.

39.18. Example. An open image of a T_2-space need not be T_2.

Indeed, let $X = \{(x, y) \in \mathbb{R}^2 : y = 0 \text{ or } y = 1\}$. Identify $(x, 0)$ with $(x, 1)$ if $x \neq 0$. The resulting quotient is $Y = \{\overline{x} = \{(x, 0), (x, 1)\} : x \neq 0\} \cup \{(0, 0)\} \cup \{(0, 1)\}$ with the quotient map $p(x, y) = \overline{x}$ if $x \neq 0$, $p((0, 0)) = (0, 0)$ and $p(0, 1) = (0, 1)$. Clearly, the map p is open. Also every neighbourhood of $(0, 0)$ contains $(-\overline{\varepsilon}, \overline{\varepsilon})$ and every neighbourhood of $(0, 1)$ contains $(-\overline{\delta}, \overline{\delta})$ for some $\varepsilon, \delta > 0$. Thus $(0, 0)$ and $(0, 1)$ do not have disjoint neighbourhood.

39.19. Example. Let X be an infinite set. Fix two symbols a, b which do not represent members of X. Let $X^* = X \cup \{a, b\}$. Define

$$\mathscr{A}_a = \{G \subset X^* : a \in G, X^* \setminus G \text{ is finite}\}.$$
$$\mathscr{A}_b = \{G \subset X^* : b \in G, X^* \setminus G \text{ is finite}\}.$$
$$\mathscr{T}_a = \mathscr{P}(X \cup \{b\}) \cup \mathscr{A}_a.$$
$$\mathscr{T}_b = \mathscr{P}(X \cup \{a\}) \cup \mathscr{A}_b.$$

Then $\mathscr{T}_a$ and $\mathscr{T}_b$ are Hausdorff topologies but $\mathscr{T}_a \cap \mathscr{T}_b$ is not Hausdorff.

Clearly, $\emptyset \in \mathscr{P}(X \cup \{b\})$, $X^* \in \mathscr{A}_a$. It is clear that each of $\mathscr{P}(X \cup \{b\})$ and $\mathscr{A}_a$ are closed under arbitrary union and finite intersection. Let $G \in \mathscr{P}(X \cup \{b\})$ and $H \in \mathscr{A}_b$. Then $X^* \setminus (G \cup H) = (X \setminus G) \cap (X^* \setminus H)$, a finite set. Thus $\mathscr{T}_a$, and similarly $\mathscr{T}_b$, is a topology on X^*.

Let $x, y \in X^*$. If $x \neq a \neq y$, then $\{x\}, \{y\}$ are disjoint open sets containing x and y respectively. Now suppose $x = a$. The $X^* \setminus \{y\}$ and $\{y\}$ are two disjoint open sets containing x and y respectively. Thus $\mathscr{T}_a$ is a Hausdorff topology. Similarly $\mathscr{T}_b$ is a Hausdorff topology.

Let $U, V \in \mathscr{T}_a \cap \mathscr{T}_b$, $a \in U$, $b \in V$. Suppose $U \cap V = \emptyset$. $U \in \mathscr{T}_a$ and $a \in U \Rightarrow X^* \setminus U$ is finite. $C \in \mathscr{T}_b$ and $b \in V \Rightarrow X^* \setminus V$ is finite. Hence $X^* = X^* \setminus (U \cap V) = (X^* \setminus U) \cup (X^* \cup V)$, a finite set. Thus X^* is finite, a contradiction. Thus a, b do not have disjoint neighbourhoods. Thus $\mathscr{T}_a \cap \mathscr{T}_b$ is not a Hausdorff topology.

39.20. Example. Consider $(\mathbb{R}, \mathscr{T}_K)$. Let $x, y \in \mathbb{R}$ be distinct points. Let $r = \frac{|x-y|}{2}$. Then $(x-r, x+r), (y-r, y+r)$ are two disjoint open sets containing x, y respectively. In a much simpler way, $\mathscr{T}_K$ is finer than the usual topology, and hence, it is a T_2-topology.

39.21. Example.

(1) A metric space is a T_2-space.
(2) The Sierpiński space is not a T_2-space.
(3) The usual topology, the lower limit topology, the upper limit topology are T_2-topologies.
(4) Indiscrete space with at least two points is not a T_2-space.
(5) A set X with any of the left order topology, the right order topology, excluded point topology, particular point topology is not a T_2-space.
(6) A set X with discrete topology is a T_2-space.
(7) A set X with the cofinite topology is a T_2-space if and only if X is finite.
(8) A set X with the cocountable topology is a T_2-space if and only if X is countable.
(9) A set X with the cosub α-topology is a T_2-space if and only if $|X| < \alpha$.
(10) A set X with the co α-topology is a T_2-space if and only if $|X| \leq \alpha$.
(11) A topology finer than a T_2-topology is also T_2.
(12) Intersection of two T_2-topologies on a set need not be T_2.

Problems

230. Let X be a Hausdorff space and $\{x_\lambda\}$ be a net in X. Show that $\{x_\lambda\}$ can converge to at most one point.

231. Show that a subspace of a T_2-space is T_2 in each of the following ways.

(1) Using definition of a T_2-space.
(2) Using convergence of nets.
(3) Using convergence of filters.

232. Let X be a set, Y be a discrete space and $f\colon X \to Y$ be a function. Show that X with the weak topology given by f is T_2 if and only if it is discrete.

40. T_3 and Regular Spaces

40.1. Definition. A topological space X is said to be a *T_3-space* if given a closed subset F of X and a point $x \in X \setminus F$, there are open sets $U, V \subset X$ such that

$$x \in U, F \subset V \text{ and } U \cap V = \emptyset.$$

A T_3-space, which is also a T_1-space, is called a *regular space*.

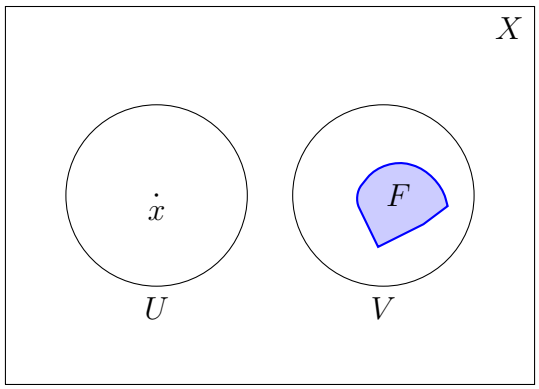


FIGURE 30. Schematic T_3-space

The following examples show that a T_3-space need not be T_2 and a T_2-space need not be T_3.

40.2. Example. A T_2-space need not be T_3.

Consider $(\mathbb{R}, \mathscr{T}_K)$. Then K is a closed set and $0 \in \mathbb{R} \setminus K$. Suppose U, V are two disjoint subsets of $\mathbb{R}$ such that $0 \in U$, $K \subset V$. Let $r > 0$ be such that $(-r, r) \setminus K \subset U$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < r$. Since $\frac{1}{n} \in V$, there is $(\frac{1}{n} - \varepsilon, \frac{1}{n} + \varepsilon) \subset V$. We can take ε small enough so that $\frac{1}{n+1} < \frac{1}{n} - \varepsilon$. Consequently, $\frac{1}{n} - \frac{\varepsilon}{2} \in U \cap V$. Thus $(\mathbb{R}, \mathscr{T}_K)$ is not T_3.

40.3. Example. An indiscrete space is trivially T_3. But it is not T_1, and hence, it is not regular.

40.4. Theorem. *A regular space is T_2.*

PROOF. Let $(X, \mathscr{T})$ be a regular space. Given two distinct points $x, y \in X$, treating $\{y\} = F$ as a closed set and x as a point out side it, we get two disjoint open sets $U, V \subset X$ such that $x \in U$ and $F \subset U$. But then $y \in V$, proving that X is T_2. □

In view of the Proposition 39.2, the following is a corollary to the above theorem.

40.5. Corollary. *A finite regular space is discrete.*

40.6. Theorem. *A topological space X is T_3 if and only if for every open set $G \subset X$ and a point $x \in G$, there exists an open set $H \subset X$ such that $x \in H \subset \overline{H} \subset G$.*

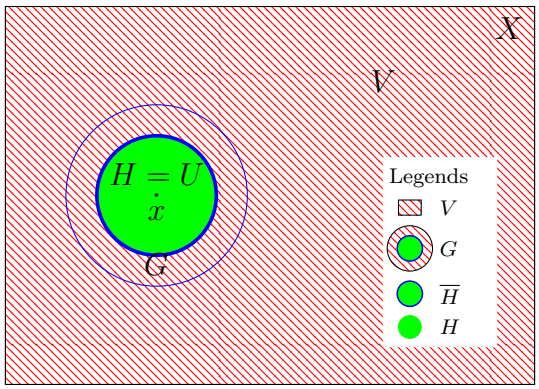


FIGURE 31. Characterization of a T_3-space

PROOF. Suppose X is T_3. Let G be an open set and $x \in G$. Let $F = X \setminus G$. Then $F \subset X$ is closed and $x \notin F$. So, there exist two open sets $U, V \subset X$ such that $x \in U$, $F \subset V$ and $U \cap V = \emptyset$. Let $H = U$. Note that $U \cap V = \emptyset$. So, $H = U \subset X \setminus V$. Since $X \setminus V$ is closed, we get $\overline{H} \subset \overline{X \setminus V} = X \setminus V$. Also, $X \setminus G = F \subset V$. So, $X \setminus V \subset G$. Thus $x \in H \subset \overline{H} \subset X \setminus V \subset G$.

Conversely, suppose for every open set $G \subset X$ and a point $x \in G$, there exists an open set $H \subset X$ such that $x \in H \subset \overline{H} \subset G$. Let $F \subset X$ be closed and $x \notin F$. Let $G = X \setminus F$. Then G is open and $x \in G$. So, there exists an open set $H \subset X$ such that $x \in H \subset \overline{H} \subset G$. Let $U = H$ and $V = X \setminus \overline{H}$. Then $x \in U$. Also $\overline{H} \subset G$. So, $F \subset X \setminus \overline{H} = V$. Since $H \subset \overline{H}$, $U \cap V = H \cap X \setminus \overline{H} = \emptyset$. Thus X is T_3. □

40.7. Corollary. *A topological space X is T_3 if and only if every point $x \in X$ has a local base consisting of closed neighbourhoods of x.*

PROOF. Suppose X is T_3, $x \in X$ and $\mathscr{B}_x$ is a local base at x. Define $\mathscr{B}'_x = \{\overline{B} : B \in \mathscr{B}_x\}$. Clearly, $\mathscr{B}'_x \subset \mathscr{U}_x$. Now let $G \in \mathscr{U}_x$. Using the T_3-condition and the Theorem 40.6, we get and open set H such that $x \in H \subset \overline{H} \subset G$. Since $\mathscr{B}_x$ is a local base at x, there is $B \in \mathscr{B}_x$ such that $x \in B \subset H$. But then $x \in \overline{B} \subset \overline{H} \subset G$. Thus $\mathscr{B}'_x$ is a local base at x consisting of closed neighbourhoods of x.

Conversely, for each $x \in X$, let $\mathscr{B}_x$ be a local base at x consisting of closed neighbourhoods of x. Let G be an open set and $x \in G$. Then there is $B \in \mathscr{B}_x$ such that $x \in B \subset G$. But then taking $H = B^\circ$, we have $x \in H \subset \overline{H} \subset B \subset G$. Thus X is T_3. □

40.8. Corollary. *A 0-dimensional space is T_3.*

PROOF. Let X be a 0-dimensional space and $\mathscr{B}$ be a base for X consisting of clopen subsets of X. Then by the Proposition 10.9, for every $x \in X$,

$$\mathscr{B}_x = \{B \in \mathscr{B} : x \in B\}$$

is a local base at x consisting of clopen subsets. Thus X is T_3. □

40.9. Theorem. *An ordered topological space is T_3, and hence, regular.*

PROOF. Let X be an ordered topological space and $x \in X$ be fixed. For $a, b \in X$ with $a < x < b$, we define

(1) c to be a point such that $a < c < x$, if it exists; otherwise $c = x$.
(2) d to be a point such that $x < d < b$, if it exists; otherwise $d = x$.

Now let G be an open set containing x.

Suppose there is a basic open set (a, b) such that $x \in (a, b) \subset G$. Let $H = \text{Int}([c, d])$.

If $a < c < x < d < b$, then $x \in (c, d) \subset \overline{(c, d)} \subset [c, d] \subset (a, b) \subset G$.

If $a < c = x < d < b$, then $[c, d)$ is open and $x \in [c, d) \subset \overline{[c, d)} \subset [c, d] \subset (a, b) \subset G$.

If $a < c < x = d < b$, then $(c, d]$ is open and $x \in (c, d] \subset \overline{(c, d]} \subset [c, d] \subset (a, b) \subset G$.

If $a < c = x = d < b$, then $\{x\} = (a, b)$, an open set. Then $H = \{x\}$ and $x \in H \subset \overline{H} = \{x\} \subset G$.

Suppose there is a basic open set (a, ∞) such that $x \in (a, \infty) \subset G$. Clearly, $x \in \text{Int}([c, \infty)) \subset \overline{\text{Int}([c, \infty))} \subset [c, \infty) \subset G$.

Suppose there is a basic open set $(-\infty, b)$ such that $x \in (-\infty, b) \subset G$. Clearly, $x \in \text{Int}((-\infty, d]) \subset \overline{\text{Int}((-\infty, d])} \subset (-\infty, d] \subset G$.

Thus X is a T_3-space. By the Corollary 38.4, X is also T_1. Thus X is regular. □

40.10. Theorem. *Every metric space is T_3.*

PROOF. Let (X, d) be a metric space. Fix $x \in X$. Since $B[x, \frac{r}{2}] \subset B(x, r)$, one easily verifies that $\mathscr{B}_x = \{B[x, r] : r > 0\}$ is a local base at x consisting of closed neighbourhoods of x. Thus X is T_3. □

40.11. Corollary. *Every metric space is regular.*

40.12. Theorem. *Subspace of a T_3-space is T_3.*

PROOF. Let X be a T_3-space and $Y \subset X$. For an $x \in Y$, let $\mathscr{B}_x$ be a local base consisting of closed neighbourhoods of x in X. Then by the Proposition 12.5, $\mathscr{B}_{xY} = \{B \cap Y : B \in \mathscr{B}_x\}$ is a local base consisting of closed neighbourhoods of x in Y. Thus Y is T_3. □

40.13. Corollary. *A subspace of a regular space is regular.*

40.14. Theorem. *Homeomorphic image of a T_3-space is a T_3-space.*

PROOF. Let X be a T_3 space and $f\colon X \to Y$ be a homeomorphism. For $x \in X$, let $\mathscr{B}_x$ be a local base at x consisting of closed neighbourhoods of x in X. But then by the Theorem 19.24, $f(\mathscr{B}_x)$ is a local base at $f(x)$ consisting of closed neighbourhoods of $f(x)$. Since f is onto, we have a local base at each point of Y consisting of its closed neighbourhoods. Thus Y is T_3. □

40.15. Corollary. *Homeomorphic image of a regular space is regular.*

40.16. Theorem. *Let $X = \prod_{i\in I} X_i$ be a product space. Then X is T_3 if and only if each X_i is T_3.*

PROOF. X_i, being homeomorphic to a subspace of X, is T_3, when X is.

Conversely, assume that X_i is T_3 for every $i \in I$. Let $U \subset X$ be an open set and $\mathbf{x} \in U$. So, there is a finite set $A \subset I$, and for each $i \in A$, an open set U_i in X_i such that $\mathbf{x} \in \cap_{i\in I}\pi_i^{-1}(U_i) \subset U$. Since X_i is T_3, for every $i \in A$, there is an open set $V_i \subset U_i$ such that $\mathbf{x}_i \in V_i \subset \overline{V_i} \subset U_i$. Hence $\mathbf{x} \in \cap_{i\in A}\pi_i^{-1}(V_i) \subset \cap_{i\in A}\pi_i^{-1}(\overline{V_i}) \subset \cap_{i\in A}\pi_i^{-1}(U_i) \subset U$. Since $\cap_{i\in A}\pi_i^{-1}(\overline{V_i})$ is closed, it follows that $\overline{\cap_{i\in A}\pi_i^{-1}(V_i)} \subset \cap_{i\in A}\pi_i^{-1}(\overline{V_i})$. Thus

$$\mathbf{x} \in \cap_{i\in A}\pi_i^{-1}(V_i) \subset \overline{\cap_{i\in A}\pi_i^{-1}(V_i)} \subset \cap_{i\in A}\pi_i^{-1}(U_i) \subset U.$$

Thus X is T_3. □

40.17. Corollary. *Product of regular spaces is regular.*

40.18. Proposition. *Let X be a T_3-space. Define $\sim_H$ on X by $x \sim_H y$ if every neighbourhood of x contains y. Then $\sim_H$ is an equivalence relation.*

PROOF. Clearly, $\sim_H$ is reflexive. Suppose $y \not\sim_H x$. Then there is an open neighbourhood U of y that does not contain x. Since X is T_3, there is an open set V in X such that $y \in V \subset \overline{V} \subset U$. Thus $X \setminus \overline{V}$ is an open set that does not contain y. Thus $x \not\sim_H$. Thus $\sim_H$ is symmetric. Now assume that $x \sim_H y$ and $y \sim_H z$. Let U be a neighbourhood of x. Then U contains y, that is, U is a neighbourhood of z. Bence U contains z. Thus $x \sim_H z$. This completes the proof. □

$\sim_H$ is an equivalence relation on a T_3 space. However, there exists a space X which is nor T_3 but $\sim_H$ is an equivalence relation on X. The following example shows the same.

40.19. Example. An infinite cofinite space X_f is not T_3, otherwise it will be regular and hence T_2, which is not the case. On the other hand let x, y be two

distinct points of X_f. Then there is an open set containing x that does not contain y. Thus $x \not\sim_H y$. Thus the relation $\sim_H = \Delta$, which is an equivalence relation.

V. N^o_tations: ☞ Let X be a T_3-space. The quotient $X/\sim_H$ is denoted by X_H and the equivalence class containing $x \in X$ is denoted by x_H. Also, q_H will denote the function $q_H : X \to X_H$ by $q_H(x) = x_H$.

40.20. Lemma. *Let X be a T_3-space. Let $U \subset X$ be open and $E \subset X$ be closed. Then $U = q_H^{-1}(q_H(U))$ and $E = q_H^{-1}(q_H(E))$.*

PROOF. Clearly, $U \subset q_H^{-1}(q_H(U))$ and $E \subset q_H^{-1}(q_H(E))$. Let $x \in q_H^{-1}(q_H(U))$. But then there is $x \in U$ such that $q_H(x) - q_H(y)$, giving $x \sim_H y$. Consequently, every neighbourhood of x also contains y. That is, $y \in U$, giving $q_H^{-1}(q_H(U)) \subset U$. Now suppose $x \in q_H^{-1}(q_H(E))$. So, there is $y \in E$ such that $x \sim_H y$. If $x \notin E$, then $x \in X \setminus E$, an open set. Hence $y \in X \setminus E$, a contradiction. This completes the proof. □

40.21. Corollary. *Let X be a T_3-space. Then q_H is an open as well as a closed map.*

PROOF. Let $U \subset X$ be open. Then $q_H^{-1}(q_H(U))$, an open set. Hence $q_H(U)$ is open. In the same way, for a closed set $E \subset X$, $q_H^{-1}(X_H \setminus q_H(E)) = q_H^{-1}(q_H(X \setminus E)) = X \setminus E$, an open set. Hence $q_H(E)$ is closed. □

40.22. Proposition. *Let X be a T_3-space. The quotient space $X/\sim_H$ is Hausdorff.*

PROOF. For $x \in X$, let x_H denote the equivalence class containing x. Suppose X_H, y_H be two distinct points of X_H. So, x, y are two distinct points of X and $x \not\sim_H y$. So, there is a neighbourhood U of x, which does not contain y. Consequently, there is an open set V such that $x \in V \subset \overline{V} \subset U$. Consequently, V and $X \setminus \overline{V}$ are disjoint open sets containing x and y respectively. Hence $q_H(U), q_H(V)$ are disjoint open sets containing x_H and y_H respectively. Thus $X_H = X/\sim_H$ is Hausdorff space. □

40.23. Definition. Let X be a T_3-space. Then $X/\sim_H$ with the quotient topology is called the *Hausdorff quotient of X*.

Problems

233. Let X be a topological space and for every $x \in X$, $\mathscr{B}_x$ be a local base at x. Show that X is T_3 if and only if for every $x \in X$ and $B \in \mathscr{B}_x$, there is $B_1 \in \mathscr{B}_x$ such that $\overline{B_1} \subset B$.

41. $T_{3\frac{1}{2}}$ and Completely Regular Spaces

41.1. Definition. A topological space X is said to be a *$T_{3\frac{1}{2}}$-space* if given a closed subset $F \subset X$ and a point $x \in X \setminus F$, there is a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(F) = \{1\}$. A $T_{3\frac{1}{2}}$-space, which is also a T_1-space, is called a *completely regular space*.

41.2. Theorem. *A subspace of $T_{3\frac{1}{2}}$-space is a $T_{3\frac{1}{2}}$-space.*

Proof. Let X be a $T_{3\frac{1}{2}}$-space and $Y \subset X$. Let $E \subset Y$ be closed and $x \in Y \setminus E$. Then there is a closed set F of X such that $F \cap Y = E$. Clearly, $x \notin F$. Hence there is a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(F) = \{1\}$. Restricting f to Y, we obtain the required function. □

41.3. Theorem. *Homeomorphic image of a $T_{3\frac{1}{2}}$-space is $T_{3\frac{1}{2}}$.*

Proof. Let X be a $T_{3\frac{1}{2}}$-space, Y be a topological space and $f\colon X \to Y$ be a homeomorphism. Let F be a closed subset of Y and $y \in Y \setminus F$. Then $f^{-1}(F)$ is a closed subset of X and $x = f^{-1}(y) \notin f^{-1}(F)$. So, there is a continuous function $g\colon X \to [0,1]$ such that $g(x) = 0$ and $g(f^{-1}(F)) = \{1\}$. Then $h = g \circ (f^{-1})$ is the required function. □

41.4. Theorem. *Let $X = \prod_{i\in I} X_i$ be a product space. Then X is $T_{3\frac{1}{2}}$ if and only if each X_i is $T_{3\frac{1}{2}}$.*

Proof. X_i, being homeomorphic to a subspace of X, is $T_{3\frac{1}{2}}$, when X is.

Conversely, suppose each X_i is $T_{3\frac{1}{2}}$, $F \subset X$ is closed and $\mathbf{x} \in X \setminus F$. Consider a basic open set $U = \cap_{k=1}^{n} \pi^{-1}(U_{i_k})$ in X, with U_{i_k} open in X_{i_k}, for each k, and $\mathbf{x} \in U \subset X \setminus F$. For every $1 \le k \le n$, let $f_k\colon X_{i_k} \to [0,1]$ be a continuous function such that $f_k(\mathbf{x}_k) = 1$ and $f_k(X_{i_k} \setminus U_{i_k}) = \{0\}$. Define $f\colon X \to [0,1]$ by $f(\mathbf{a}) = \min\{(\pi_{i_k} \circ f_k)(\mathbf{a}) : 1 \le k \le n\}$, $(\mathbf{a} \in X)$. Clearly, being minimum of finitely many continuous functions, f is a continuous function. One verifies that f satisfies the required properties. □

41.5. Theorem. *A $T_{3\frac{1}{2}}$ is a T_3-space.*

Proof. Let X be a $T_{3\frac{1}{2}}$ space. Let $F \subset X$ be closed and $x \in X \setminus F$. Since X is $T_{3\frac{1}{2}}$, there is a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and

$f(F) = \{1\}$. Let $U = f^{-1}([0, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, 1])$. Then U, V are open because f is continuous. Also, $x \in U, F \subset V$, and $U \cap V = \emptyset$, completing the proof. □

The following is at once now.

41.6. Theorem. *A completely regular space is regular, and hence, T_3, T_2, T_1 and T_0.*

41.7. Theorem. *A metric space is completely regular.*

Proof. Since every metric space is T_1, it is sufficient to show that a metric space is $T_{3\frac{1}{2}}$. Let (X, d) be a metric space, $F \subset X$ be closed and $x_0 \in X \setminus F$. Then $f\colon X \to [0, 1]$ defined by $f(x) = \frac{d(x,x_0)}{d(x,x_0)+d(x,F)}$, $(x \in X)$, is a continuous function. Also, $f(x_0) = 0$ and $f(F) = \{1\}$. □

41.8. Theorem. *A 0-dimensional space is $T_{3\frac{1}{2}}$.*

Proof. Let X be a 0-dimensional space with a base $\mathscr{B}$ consisting of clopen sets. Let $x \in X$ and F be a closed set such that $x \notin F$. Thus $x \in X \setminus F$, an open set. Hence there is $B \in \mathscr{B}$ such that $x \in B \subset X \setminus F$. Since B is a clopen set, χ_B, the characteristic function of B is continuous. Let $f = 1 - \chi_B$, Then $f(x) = 0$, $f(F) = \{1\}$, Thus X is a $T_{3\frac{1}{2}}$-space. □

The following example is from [Dan12]. In fact, we are aiming first to produce an example of a regular space which is not completely regular.

41.9. Example. Let $S = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$. For each $x \in \mathbb{R}$, let

$$V_x = \{(x, y) : 0 \leq y \leq 2\} = \{x\} \times [0, 2].$$
$$D_x = \{(s, s - x) : x \leq s \leq x + 2\}.$$
$$O_x = V_x \cup D_x.$$

Now define

$$\mathscr{B}_1 = \{\{(x, y)\} : y > 0\}.$$
$$\mathscr{B}_2 = \{O_x \setminus F : x \in \mathbb{R}, x \notin F \subset O_x \text{ is finite}\}.$$
$$\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2.$$

Then $\mathscr{B}$ is a base for a topology on S. Indeed, the union of $\mathscr{B}$ is S and $\mathscr{B}$ is closed under finite intersection.

We also prove that each $B \in \mathscr{B}_1 \cup \mathscr{B}_2$ is clopen.

Let $B = \{(x, y)\} \in \mathscr{B}_1$ and $(x_0, y_0) \in S \setminus B$. If $y_0 \neq 0$, then $\{(x_0, y_0)\}$ is an open set not intersecting B. If $y_0 = 0$, consider the open set $O_x \setminus F$, where F is a finite set containing (x, y) if $(x, y) \in O_x$.

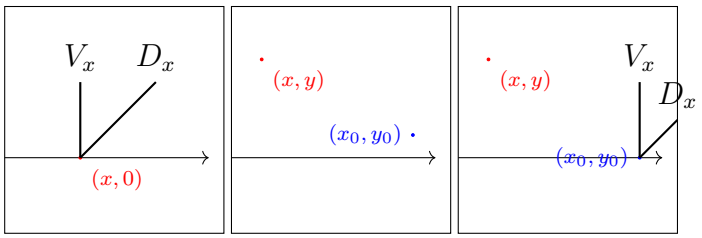


FIGURE 32. Plane and Point

Let $B = O_x \setminus F$, where $F \subset O_x$ is finite and $(x_0, y_0) \in S \setminus B$. If $y_0 > 0$, then $\{(x_0, y_0)\}$ is the open set not intersecting B. Let $y_0 = 0$. Then $O_{x_0} \cap B = F$ is a finite set. Define $B_{x_0} = O_{x_0} \setminus F$. Then $B_{x_0} \cap B = \emptyset$. Thus S is 0-dimensional, and hence, a $T_{3\frac{1}{2}}$-space. One easily verifies that for every real number x, $\cap\{O_x \setminus F : F \subset O_x \setminus \{x\}\} = \{x\}$, and hence, S is T_1. Thus S is a completely regular space.

41.10. Example. Let S be as in the Example 41.9 and $T = S \cup \{(0, -1)\}$. For every $n \in \mathbb{N}$, let $U_n = \{(x, y) : x \geq n, y \geq 0\}$. Let T carry the topology determined by the basic open sets $\mathscr{B}_1 \cup \mathscr{B}_2 \cup \mathscr{B}_3$ where

$$\mathscr{B}_3 = \{U_n \cup \{(0, 1)\} : n \in \mathbb{N}\}.$$

The following lemmas show that T is regular but not completely regular. In the following discussion till the end of this section, H_n will stand for the set $[n, n + 1] \times \{0\}$ and $H = \mathbb{R} \times \{0\}$.

41.11. Lemma. *T is regular.*

PROOF. Let $B \in \mathscr{B}_1 \cup \mathscr{B}_2$ and $x \in B$. Then $x \in B \subset \overline{B} \subset B$. Let $B = U_n \cup \{(0, 1)\}$. Clearly, $(0, 1) \in U_{n+2} \subset \overline{U_{n+2}} \subset U_n$. Thus T is T_3. Since S is an open T_1-subspace of T, T is T_1. □

41.12. Lemma. *Let $f : S \to \mathbb{R}$ be a continuous function and $(a, 0) \in S$. If $f((a, 0)) = 0$, then $f(x) = 0$ for all but countably many $x \in O_a$.*

PROOF. For $n \in \mathbb{N}$, $G = f^{-1}(-\frac{1}{n}, \frac{1}{n}) \in \mathscr{U}_{(a,0)}$. Let $B = O_a \setminus F \in \mathscr{B}_2$ be such that $B \subset G$. Thus $A_n = \{x \in O_a : |f(x)| \geq \frac{1}{n}\} \subset F$, and hence, is finite. Thus

$$\{x \in O_a : f(x) \neq 0\} = \cup A_n,$$

which is at most countable. □

41.13. Lemma. *Let $f : S \to \mathbb{R}$ be a continuous function and $(a, 0) \in S$. If $f(x) = 0$ for infinitely many $x \in O_a$, then $f((a, 0)) = 0$.*

PROOF. Suppose $f((a, 0)) = r \neq 0$. Let $W = (r - \frac{|r|}{2}, r + \frac{|r|}{2})$, Then there is $F \subset O_a$ such that $f(O_a \setminus F) \subset W$. Thus only finitely many $x \in O_a$ are such that $f(x) = 0$, a contradiction. Thus $f((a, 0)) = 0$. □

41.14. Lemma. *Let $n, m \in \mathbb{N}$ and $n \leq m$. Let $f: T \to \mathbb{R}$ be a continuous function. Suppose $f((x,0)) = 0$ for infinitely many points $(x,0) \in H_n$, then $f((x,0)) = 0$ for all but countably many points in H_m.*

PROOF. It is sufficient to prove for $m = n+1$. Let $A = \{(x_k, 0) : k \in \mathbb{N}\} \subset H_n$ be such that $f(A) = \{0\}$. By the Lemma 41.12, for each $k \in \mathbb{N}$, there is at most a countable subset $A_k = \{w \in D_{x_k} : f(w) \neq 0\}$. Let $B = \cup_{k=1}^{\infty}\{\pi_1(w) : w \in A_k\}$, where $\pi_1(x,y) = x$, $((x,y) \in S)$. Let $C = [n+1, n+2] \setminus C$. Let $(x,0) \in C$. Then for each $k \in \mathbb{N}$, $D_{x_j} \cap V_x$ is singleton, say $D_{x_k} \cap V_x = \{b_k\}$. But then $f(b_k) = 0$. Thus there is a countably infinite set $E = \{b_k : k \in \mathbb{N}\} \subset V_x$ such that $f(E) = \{0\}$. Hence by the Lemma 41.13, $f(x,0) = 0$. Thus $C \subset H_{n+1}$ and $f(C) = \{0\}$. Since C contains all but countably many points of H_{k+1}, the result follows. □

41.15. Lemma. *T is not $T_{3\frac{1}{2}}$.*

PROOF. Consider the closed set $F = H_0 = [0,1] \times \{0\}$ and $x = \{(0,-1)\}$. Suppose there is a function $f: T \to [0,1]$ such that $f(F) = \{0\}$ and $f((0,-1)) = 1$. But then $f^{-1}((0.5,1]) \in \mathscr{U}_{(0,-1)}$. So $H_n \subset f^{-1}((0.5,1])$ for some n. But then $0 \notin f(H_n) \subset (0.5,1]$ and $f(H_0) = \{0\}$, a contradiction to the Lemma 41.14. Thus T is not $T_{3\frac{1}{2}}$. □

Now we characterize the $T_{3\frac{1}{2}}$-spaces.

41.16. Theorem. *A topological space is $T_{3\frac{1}{2}}$ if and only if its topology is the weak topology given by all real-valued bounded continuous functions.*

PROOF. Suppose X is a $T_{3\frac{1}{2}}$-space. Let U be an open set and $x \in U$. So, there is a continuous function $f: X \to [0,1]$ such that $f(x) = 0$ and $f(y) = 1$ whenever $y \notin U$. Thus $x \in f^{-1}([0, \frac{1}{2})) \subset U$. Thus U is open in the weak topology given by all real-valued bounded continuous functions on X. Clearly, since every real-valued bounded continuous function is continuous in the given topology, the weak topology is weaker than the given topology.

Conversely, suppose X carries the weak topology given by all real-valued bounded continuous functions. Let $F \subset X$ be closed and $a \in X \setminus F = U$. Since U is open, there are bounded continuous functions $f_1, f_2, \ldots, f_n$ and bounded open intervals $(a_i, b_i) \subset \mathbb{R}$ such that $a \in \cap_{i=1}^{n} f_i^{-1}((a_i, b_i)) \subset U$. For $1 \leq i \leq n$, define

(1) $g_i = -f_i$,
(2) $h_i = g_i + b_i$,
(3) $k_i = f_i - a_i$,
(4) $p_i = \max\{h_i, 0\}$,
(5) $q_i = \max\{k_i, 0\}$, and
(6) $f = p_1 p_2 \cdots p_n q_1 q_2 \cdots q_n$.

Note here that each of $p_i, q_i : X \to [0,\infty)$, and so, $f : X \to [0,\infty)$ is a bounded continuous function. Also,

$$\begin{aligned}\cap_{i=1}^n f_i^{-1}((a_i,b_i)) &= \cap_{i=1}^n [f_i^{-1}((-\infty,b_i)) \cap f_i^{-1}((a_i,\infty))] \\ &= \cap_{i=1}^n [g_i^{-1}((-b_i,\infty)) \cap f_i^{-1}((a_i,\infty))] \\ &= \cap_{i=1}^n [h_i^{-1}((0,\infty)) \cap k_i^{-1}((0,\infty))] \\ &= \cap_{i=1}^n [p_i^{-1}((0,\infty)) \cap q_i^{-1}((0,\infty))].\end{aligned}$$

Suppose $y \in X$ such that $f(y) > 0$. So, for every i, $p_i(y), q_i(y) \in (0,\infty)$. So $y \in U$. Thus if $y \notin U$, then $f(y) = 0$. Thus $f(F) = \{0\}$. Since $p_i(x), q_i(x) > 0$ for every i, $f(x) > 0$. Let $f(X) \subset [0,c]$ and $0 < d = f(x) \le c$. Define $r : [0,c] \to [0,1]$ by $r(t) = \begin{cases} \frac{t}{d}, & \text{if } t \le d; \\ 1, & \text{if} t \ge d, \end{cases}$ and $g = 1 - r \circ f$. Then one easily verifies that $g : X \to [0,1]$ and $g(x) = 0$, $g(F) = \{1\}$. This completes the proof. □

VI. N$^o{}_t$ations: ☞ $C_b(X,\mathbb{R})$ = the set of all real-valued bounded continuous functions on X.

41.17. Definition. Let I be a set. The product space $[0,1]^I$ is called a *cube*. Also for $n \in \mathbb{N}$, let $I_n = [0, \frac{1}{n}]$. Then $\prod_{n=1}^{\infty} I_n$ is called the *Hilbert cube*.

41.18. Corollary. *A topological space is completely regular if and only if it is homeomorphic to a subspace of a cube $[0.1]^I$ for some set I.*

Proof. If $X \subset [0,1]^I$, then as a subspace of a completely regular space, X is completely regular. Conversely, suppose that X is completely regular. Then X has the weak topology induced by $C_b(X,\mathbb{R})$. For $f \in C_b(X,\mathbb{R})$, let J_f be a closed and bounded interval such that $f(X) \subset J_f$. Then the evaluation map of $C_b(X,\mathbb{R})$ to $\prod_{f \in C_b(X,\mathbb{R})} J_f$, $e : X \to \prod_{f \in C_b(X,\mathbb{R})} J_f$, defined by $e(x)_f = f(x)$, $(x \in X, f \in C_b(X,\mathbb{R}))$, is an embedding. Since each J_f is homeomorphic to $[0,1]$, we conclude that $\prod_{f \in C_b(X,\mathbb{R})} J_f$ is homeomorphic to $[0,1]^{C_b(X,\mathbb{R})}$. □

42. T_4 and Normal Spaces

42.1. Definition. A topological space X is said to be a *T_4-space* if given disjoint closed subset $E, F \subset X$, there are open sets $U, V \subset X$ such that

$$E \subset U, F \subset V \text{ and } U \cap V = \emptyset.$$

A T_4-space, which is also a T_1-space, is called a *normal space*.

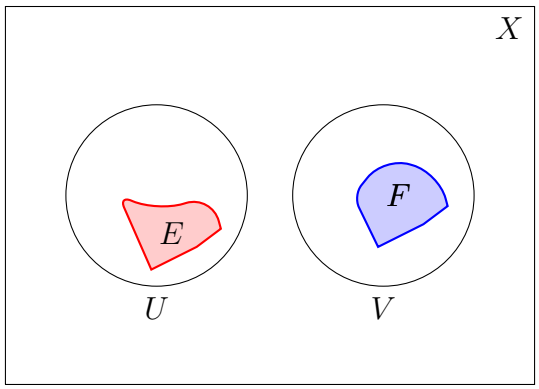


FIGURE 33. Schematic T_4-space

42.2. Example. An indiscrete space is trivially T_4. But it is not T_1, and hence, it is not normal.

42.3. Theorem. *A normal space is regular.*

PROOF. Let $(X, \mathscr{T})$ be a regular space. Let $F \subset X$ be closed and $x \in X \setminus F$. Since X is T_1, $E = \{x\}$ is closed. Also, $E \cap F = \emptyset$. Thus there are two disjoint open sets $U, V \subset X$ such that $E \subset U, F \subset U$. But then $x \in U$, $F \subset V$, proving that X is T_3. Since a normal space is T_1, X is regular. □

42.4. Example. A T_4-space need not be T_3.

Consider the real line with the topology consisting of the empty set and sets of the form (a, ∞) $(a \in \mathbb{R})$. Then there are no disjoint closed sets. So, $\mathbb{R}$ is vacuously, T_4. On the other hand, $(-\infty, 0]$ is a closed set not containing 1 and these two cannot be separated by means of two disjoint open sets as no two nonempty open subsets of $\mathbb{R}$ are disjoint.

42.5. Theorem. *A topological space X is T_4 if and only if for every open set $G \subset X$ and a closed set E such that $E \in G$, there exists an open set $H \subset X$ such that $F \subset H \subset \overline{H} \subset G$.*

PROOF. Suppose X is T_4. Let G be an open set and E be a closed set such that $E \subset G$. Let $F = X \setminus G$. Then $F \subset X$ is closed and $E \cap F$. So there exist two open sets $U, V \subset X$ such that $E \subset U$, $F \subset V$ and $U \cap V = \emptyset$. Let $H = U$. Note that $U \cap V = \emptyset$. So, $H = U \subset X \setminus V$. Since $X \setminus V$ is closed, we get $\overline{H} \subset \overline{X \setminus V} = X \setminus V$. Also, $X \setminus G = F \subset V$. So, $X \setminus V \subset G$. Thus $E \subset H \subset \overline{H} \subset X \setminus V \subset G$.

Conversely, suppose for every open set $G \subset X$ and a closed set E with $E \subset G$, there exists an open set $H \subset X$ such that $E \subset H \subset \overline{H} \subset G$. Let $E, F \subset X$ be disjoint and closed. Let $G = X \setminus F$. Then G is open and $E \subset G$. So, there exists an open set $H \subset X$ such that $E \subset H \subset \overline{H} \subset G$. Let $U = H$ and $V = X \setminus \overline{H}$. Then $E \subset U$. Also $\overline{H} \subset G$. So, $F \subset X \setminus \overline{H} = V$. Since $H \subset \overline{H}$, $U \cap V = H \cap X \setminus \overline{H} = \emptyset$. Thus X is T_4. □

42.6. Theorem. *Every metric space is T_4.*

PROOF. Let (X, d) be a metric space and E, F be two disjoint closed subsets of X. Define $f\colon X \to [0,1]$ be $f(x) = \frac{d(x,E)}{d(x,E)+d(x,F)}$, $(x \in X)$. Then by the Corollary 28.7, f is continuous. Let $U = f^{-1}[0, 0.5)$ and $V = f^{-1}((0.5, 1])$. Then U, V are open. Also $x \in E \Rightarrow d(x, E) = 0 \Rightarrow f(x) = 0$; so that $x \in U$. Thus $E \subset U$. Similarly, $F \subset V$. Thus X is T_4. □

42.7. Corollary. *Every metric space is normal.*

42.8. Theorem. *Homeomorphic image of a T_4-space is a T_4-space.*

PROOF. Let X be a T_4 space and $f\colon X \to Y$ be a homeomorphism. Let $E, F \subset Y$ be disjoint and closed. Then $f^{-1}(E), f^{-1}(F)$ are disjoint closed subsets of X. So, there exist two disjoint open sets $U, V \subset X$ such that $f^{-1}(E) \subset U$ and $f^{-1}(F) \subset V$. But a homeomorphism is an open map so that $f(U), f(V)$ are open in Y, $E \subset f(U), F \subset f(V)$. Since f is one-one, $f(U) \cap f(V) = f(U \cap V) = \emptyset$. Thus Y is T_4. □

42.9. Corollary. *Homeomorphic image of a normal space is normal.*

42.10. Theorem. *A closed subspace of a T_4-space is T_4.*

PROOF. Let $(X, \mathscr{T})$ be a T_4-space and $Y \subset X$ be closed. Let E, F be two disjoint closed subsets of Y. Then by the Proposition 13.10, E, F are closed subsets of X. Using the fact that X is T_4, we get two disjoint open sets U, V in X such that $E \subset U, F \subset V$. But then $U_1 = U \cap Y$ and $V_1 = V \cap Y$ are two disjoint open subsets of Y such that $E \subset U_1, F \subset V_1$. Thus Y is T_4. □

42.11. Corollary. *A closed subspace of a normal space is normal.*

In general, a subspace of a normal space need not be normal. Hence subspace of a T_4-space need not be T_4. Example of a T_4-space whose subspace is not T_4 is easy. In fact, well within finite topological spaces we get such an example. However subspace of a finite normal space is a normal space. Indeed a finite normal space is discrete and every subspace of discrete space is discrete, and hence, normal. So, in order to find an example of a normal space whose subspace is not normal, we have to consider an infinite space.

42.12. Example. Subspace of a T_4-space need not be T_4.

Indeed let $X = \{a, b, c, d\}$. Consider the particular point topology,

$\mathscr{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$.

Consider the excluded point topology,

$$\mathscr{T}_2 = \{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{b\}, \{c\}, \{b,c\}\}.$$

Let

$$\mathscr{T} = \mathscr{T}_1 \cap \mathscr{T}_2 = \{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}.$$

Then all nonempty $\mathscr{T}$-closed sets contain d. Thus we do not have two nonempty disjoint closed subsets. So, X is T_4. Let $Y = \{a,b,c\}$. Then

$$\mathscr{T}_Y = \{\emptyset, Y, \{a\}, \{a,b\}, \{a,c\}\}.$$

Then $\{b\}, \{c\}$ are disjoint closed subsets of Y. But Y does not have disjoint nonempty open subsets. Thus Y is not T_4.

42.13. Lemma. *Let X be a T_4-space, $Y, Z \subset X$. Suppose Y is closed and discrete and Z is dense. Then $|Y| < 2^{|Z|}$.*

Proof. Let $E \subset Y$. Since Y is discrete, $E, Y \setminus E$ are closed disjoint subsets of Y. Since Y is closed, $E, Y \setminus E$ are closed in X. So, there are two disjoint open sets $U(E), V(E) \subset X$ such that $E \subset U(E), (Y \setminus E) \subset V(E)$. Define $f\colon \mathscr{P}(Y) \to \mathscr{P}(Z)$ by $f(E) = U(E) \cap Z$, $(E \in \mathscr{P}(Y))$. Suppose $E, F \subset Y$ and $E \setminus F \neq \emptyset$. Then $U(E) \cap V(F) \neq \emptyset$. Since Z is dense in X and $U(E) \cap V(F)$ is a nonempty open subset of X, $U(E) \cap V(F) \cap Z \neq \emptyset$. But then $U(E) \cap V(F) \cap Z \subset U(E) \cap Z$. Let $x \in E \setminus F$. Then $x \in U(E)$. Also $x \in Y \setminus E \subset V(F)$. Thus $x \in U(E) \cap V(F) \cap Z$ but $x \notin U(F) \cap Z$. Thus $f(E) = U(E) \cap Z \neq U(F) \cap Z = f(F)$. That is, f is one-one. Hence $|Y| < 2^{|Y|} \leq 2^{|Z|}$. □

42.14. Lemma. *$X = \mathbb{R}_\ell \times \mathbb{R}_\ell$ is not T_4.*

Proof. Let $Z = \mathbb{Q} \times \mathbb{Q}$. Then any basic open set $B = [a,b) \times [c,d)$ contains infinitely may points of Z. Thus $\overline{Z} = X$.

Let $Y = \{(x, -x) : x \in \mathbb{R}\}$. For $x \in \mathbb{R}$, $\{(x,-x)\} = Y \cap [x, x+1) \times [-x, 1-x)$. Thus Y is discrete. Also, Y is closed. Since $|Y| = |\mathbb{R}| = \mathfrak{c} = 2^{|Z|} = 2^{|\aleph_0|}$, $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not T_4. □

42.15. Example. The topological space S defined in the Example 41.9 is not normal. Indeed again $Z = S$ is dense in S and $Y = \mathbb{R} \times \{0\}$ is discrete closed subspace with same cardinality.

43. Characterizations of a T_4-space

The following Lemma uses a simple fact that $\frac{1}{2^n} \to 0$. Recall that a *dyadic rational number* is a rational number of the form $\frac{p}{2^n}$, $p \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}$.

43.1. Lemma. *$A = \{\frac{k}{2^n} : n \in \mathbb{N}, 0 \leq k \leq 2^n\}$ is dense in $[0,1]$.*

PROOF. Let $(a,b) \subset [0,1]$. Let $\varepsilon = (b-a)$. Since $\frac{1}{2^n} \to 0$, there is $m \in \mathbb{N}$ such that $\frac{1}{2^m} < \varepsilon$. Let $B = \{k \in \mathbb{N} \cup \{0\} : \frac{k}{2^m} \leq a\}$. Let $t = \sup B$. Note that $0 \in B$, and B is finite so that $\frac{t}{2^m} \leq a$. If $\frac{t+1}{2^m} > b$, then $\varepsilon = b - a < \frac{1}{2^m}$, a contradiction. Thus $a < \frac{t+1}{2^m} < b$. This completes the proof. □

The following characterizes a T_4 space in terms of separation by means of a continuous functions.

43.2. Theorem. (Urysohn's Lemma) *Let X be a topological space. Then X is T_4-space if and only if for every pair of disjoint closed subsets A, B of X, there is a continuous function $f\colon X \to [0,1]$ such that $f(A) = \{0\}$, $f(B) = \{1\}$.*

PROOF. The first step is to inductively construct a family of open subsets of X. Define $U_1 = X \setminus B$. Let $m = 1$. Choose an open set $U_{\frac{1}{2}}$ such that $A \subset U_{\frac{1}{2}} \subset \overline{U_{\frac{1}{2}}} \subset U_1$. In the next step, for $m = 2$, we construct $U_{\frac{1}{2^2}}, U_{\frac{3}{2^2}}$ such that

$$A \subset U_{\frac{1}{2^2}} \subset \overline{U_{\frac{1}{2^2}}} \subset U_{\frac{1}{2}} \subset \overline{U_{\frac{1}{2}}} \subset U_{\frac{3}{2^2}} \overline{U_{\frac{3}{2^2}}} \subset U_1$$

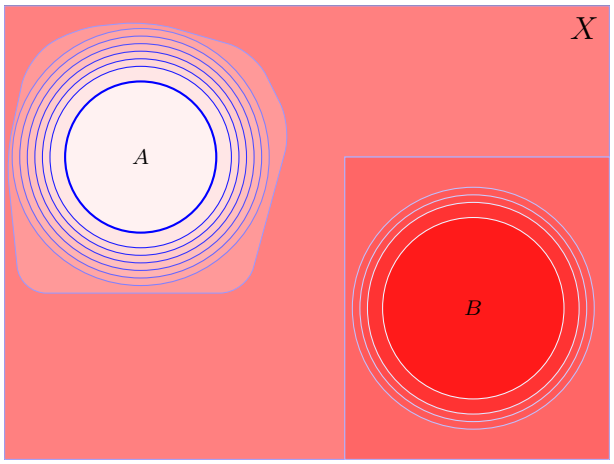


FIGURE 34. Urysohn Function

Having constructed $U_{\frac{1}{2^{m-1}}}, U_{\frac{3}{2^{m-1}}}, \ldots, U_{\frac{2^{m-1}-1}{2^{m-1}}}$, inductively, we construct $U_{\frac{1}{2^m}}, U_{\frac{3}{2^m}}, \ldots, U_{\frac{2^m-1}{2^m}}$ so that

$$A \subset U_{\frac{1}{2^m}} \subset \overline{U_{\frac{1}{2^m}}} \subset U_{\frac{1}{2^{m-1}}} \subset \overline{U_{\frac{1}{2^{m-1}}}} \subset U_{\frac{3}{2^m}} \overline{U_{\frac{3}{2^m}}} \cdots \subset \overline{U_{\frac{2^m-1}{2^{m-1}}}} \subset U_{\frac{2^m-1}{2^m}} \overline{U_{\frac{2^m-1}{2^m}}} \subset U_1$$

Thus for every dyadic rational number $0 \leq r \leq 1$, we have constructed U_r such that $0 \leq r \leq s \leq 1 \Rightarrow A \subset \overline{U_r} \subset U_s \subset \overline{U_s} \subset U_1$. Define $f\colon X \to [0,1]$ by

$$f(x) = \begin{cases} 1, & \text{if } x \notin U_1; \\ \inf\{r : x \in U_r\}, & \text{if } x \in U_1. \end{cases}$$

Let $x \in X$. Suppose $f(x) = 0$ and $[0,a) \subset [0,1]$. Then let $r \in [0,a)$ be a dyadic rational number. Then $x \in U_r$ and $f(U_r) \subset [0,r] \subset [0,a)$. Suppose $f(x) = 1$ and $(a,1] \subset [0,1]$. let $r \in (a,1]$ be a dyadic rational number. Then $x \in X \setminus \overline{U_r}$ and $f(X \setminus \overline{U_r}) \subset [r,1] \subset (a,1]$. Suppose $f(x) \in (a,b) \subset [0,1]$. Let $r, s \in (a,b)$ be dyadic rational numbers such that $a < r < f(x) < s < b$. Then $x \in U_s \setminus \overline{U_r}$

and $f(U_s \setminus \overline{U_r}) \subset (r,s) \subset (a,b)$. Thus f is continuous. Clearly, $f(A) = \{0\}$ and $f(B) = \{1\}$. This completes the proof.

Conversely, let A, B be two disjoint closed subsets of X. Let $f\colon X \to [0,1]$ be a continuous function such that $f(A) = \{0\}, f(B) = \{1\}$. Then $U = f^{-1}([0, 0.5))$ and $V = f^{-1}((0.5, 1])$ are disjoint open subsets of X such that $A \subset U$, $B \subset V$. □

Now the following is at once because singletons are closed in a normal space.

43.3. Corollary. *A normal space is completely regular.*

In view of the above corollary and the Theorem 41.2, the following is at once.

43.4. Corollary. *A subspace of a normal space is completely regular.*

43.5. Theorem. (Tietze's Extension Theorem) *A topological space X is T_4 if and only if for every closed subset $A \subset X$, every continuous function $f\colon A \to [-1,1]$ admits a continuous extension to whole of X*

Proof. Suppose for every closed subset A of X, every continuous function $f\colon A \to [-1,1]$ admits a continuous extension to whole of X. Let E, F be two disjoint closed subsets of X. Clearly, $\chi_F\colon E \cup F \to [-1,1]$ is continuous, and hence, it admits a continuous extension f to whole of X. But then by the Urysohn's Lemma, X is T_4.

Conversely, suppose X is T_4. Let $A \subset X$ be closed and $f\colon A \to [-1,1]$ be continuous. Let

$$E_1 = \{x \in A : f(x) \le -\tfrac{1}{3}\} \text{ and } F_1 = \{x \in A : f(x) \ge \tfrac{1}{3}\}$$

and $f_1\colon X \to [-\frac{1}{3}, \frac{1}{3}]$, a continuous function, such that $f_1(E_1) = \{-\frac{1}{3}\}$ and $f_1(F_1) = \{\frac{1}{3}\}$. Let $g_1 = f - f_1$ on A. Then clearly, for $x \in A$,

$$|g_1(x)| = |f(x) - f_1(x)| \le \tfrac{2}{3}$$

Again let

$$E_2 = \{x \in A : g_1(x) \le -\tfrac{2}{3^2}\} \text{ and } F_2 = \{x \in A : g_1(x) \ge \tfrac{2}{3^2}\}$$

and $f_2\colon X \to [-\frac{2}{3^2}, \frac{2}{3^2}]$, a continuous function, such that $f_2(E_2) = \{-\frac{2}{3^2}\}$ and $f_2(F_2) = \{\frac{2}{3^2}\}$. Let $g_2 = g_1 - f_2 = f - f_1 - f_2$ on A. Then clearly, for $x \in A$,

$$|g_2(x)| = |f(x) - f_1(x) - f_2(x)| \le (\tfrac{2}{3})^2$$

Continuing inductively, for every n, we get two disjoint closed subsets $E_n, F_n \subset A$, and hence, of X, a continuous function $f_n\colon X \to [-\frac{2^{n-1}}{3^n}, \frac{2^{n-1}}{3^n}]$ such that $f(E_n) = \{-\frac{2^{n-1}}{3^n}\}$, $f(F_n) = \{\frac{2^{n-1}}{3^n}\}$. taking $g_n = g_{n-1} - f_n$ on A, we have for $x \in A$,

$$|g_n(x)| = |f(x) - f_1(x) - f_2(x) - \cdots - f_n(x)| \le (\tfrac{2}{3})^n$$

Define $F\colon X \to [-1,1]$ by $F(x) = \sum_{n=1}^{\infty} f_n(x)$, $(x \in X)$. Then clearly, for $x \in A$, $f(x) - F(x) = f - \sum_{n=1}^{\infty} f_n(x) = 0$.

To prove the continuity of f on X, let $x \in X$ and $\varepsilon > 0$ be given. Let $N \in \mathbb{N}$ be such that $\sum_{n=N+1}^{\infty}(\frac{2}{3})^n \leq \frac{\varepsilon}{2}$. Using the continuity of f_n, choose $U_n \in \mathscr{U}_x$ such that $y \in U_n \Rightarrow |f_n(x) - f_n(y)| \leq \frac{\varepsilon}{2N}$ for $n = 1, 2, 3, \ldots, N$ and let $U = \cap_{n=1}^{N} U_n$. Then

$$|F(x) - F(y)| \leq \sum_{n=1}^{N} |f_n(x) - f_n(y)| + \sum_{n=N+1}^{\infty} (\tfrac{2}{3})^n < \varepsilon.$$

This completes the proof. □

43.6. Remark. There is nothing great about $[-1, 1]$. In the notation of the Tietze's extension theorem,

(1) Let $f\colon A \to \mathbb{R}$. Let $h\colon \mathbb{R} \to (-1, 1)$ be a homeomorphism. Then $k = h \circ f\colon A \to (-1, 1) \subset [-1, 1]$ is a continuous function. Considering $k\colon A \to [-1, 1]$, and applying Tietze's extension theorem, we get a continuous extension $F\colon X \to [-1, 1]$ of k. Let $B = F^{-1}(\{-1, 1\})$. Then B is closed and $A \cap B = 0$. Let $H\colon X \to [0, 1]$ be a continuous function such that $H(B) = \{0\}$ and $H(A) = \{1\}$. Then $G = F \cdot H\colon X \to (-1, 1)$ is a continuous function and $H(x) = F(x)H(x) = F(x) = k(x)$, if $x \in A$. Now let $K = h^{-1} \circ H$. $K(x) = h^{-1}(H(x)) = h^{-1}(k(x)) = h^{-1}(h(f(x))) = f(x)$ for every $x \in A$. Thus K continuously extends f.

(2) Applying the same trick as in the above, we can continuously extend any continuous function $f\colon A \to B$, where B is any of $[a, b], (a, b), (a, \infty), (-\infty, a)$.

Problems

234. Show that a topological space is T_4 if and only if given two disjoint closed subsets E, F of X and real numbers $a < b$, there is a continuous function $f\colon X \to [a, b]$ such that $f(A) = \{a\}$ and $f(B) = \{b\}$.

CHAPTER 8

Countability and Compactness

In this chapter we shall revisit the countability axioms. Recall the Definition 11.1 that a topological space is *first countable* if it has a countable local base at each of its points and is *second countable* if it has a countable base. As the Theorem 11.3 states, every second countable space is first countable. However an uncountable discrete space is first countable but not second countable.

44. First and Second Countable Spaces

44.1. Theorem. *A subspace of a first countable space is first countable.*

PROOF. Let X be a first countable space and $Y \subset X$. Let $\mathscr{B}_x$ be a countable local base at x. Then $\mathscr{B}_{xY} = \{B \cap Y : B \in \mathscr{B}_x\}$ is a countable local base at x in Y. Thus Y is first countable. □

44.2. Theorem. *A subspace of a second countable space is second countable.*

PROOF. Let X be a second countable space and $Y \subset X$. Let $\mathscr{B}$ be a countable base for X. Then $\mathscr{B}_Y = \{B \cap Y : B \in \mathscr{B}\}$ is a countable base for Y. Thus Y is second countable. □

An arbitrary product of first countable spaces need not be first countable.

44.3. Example. Let $I = \mathbb{R}$ and for every $i \in I$, let $X_i = \mathbb{R}$. Then $\mathbb{R}^{\mathbb{R}} = \prod_{i \in I} X_i$ is an uncountable product of first countable spaces. We show that it is not first countable. Suppose $B_{\mathbf{0}} = \{B_n : n \in \mathbb{N}\}$ is a local base at $\mathbf{0}$ in $\mathbb{R}^{\mathbb{R}}$. Then for every $n \in \mathbb{N}$, $A_n = \{r \in \mathbb{N} : \pi_r(B_n) \neq \mathbb{R}\}$ is finite. Consequently, $\mathbf{x} \in \mathbb{R}^{\mathbb{R}}$ defined as

$$\mathbf{x}_r = \begin{cases} 0, & \text{if } r \in \cup A_n; \\ 1, & \text{if } r \notin \cup A_n, \end{cases}$$

is in every B_n. Thus $\mathbb{R}^{\mathbb{R}}$ is not T_1, a contradiction. Thus $\mathbb{R}^{\mathbb{R}}$ is not first countable.

The following Example is at once now.

44.4. Example. A product of second countable spaces need not be second countable.

However we have the following.

44.5. Theorem. *Let $\{X_i : i \in I\}$ be a family of topological spaces. Then $X = \prod_{i\in I} X_i$ is first countable if and only if each X_i is first countable and all but countably many X_i are indiscrete spaces.*

PROOF. Suppose that X is first countable. Since each X_i is homeomorphic to a subspace of X, and since subspace and homeomorphic image of a first countable space are first countable, it follows that each X_i is first countable.

Let $K = \{i \in I : X_i \text{ is not } \textit{indiscrete}\}$. For each $i \in K$, choose a nonempty proper subset $U_i \in \mathscr{T}_i$. For $i \notin K$, let $U_i = X_i$. By axiom of choice, let $\mathbf{x} \in X$ be such that $\mathbf{x}_i \in U_i$ for every $i \in I$. Let $\mathscr{B}_\mathbf{x} = \{B_n : n \in \mathbb{N}\}$ be a local base at $\mathbf{x}$. Clearly, $I_n = \{i \in I : \pi_i(B_n) \neq X_i\}$ is finite, and hence, $J = \cup I_n$ is at most countable. Fix $p \in K$, then $\mathbf{x} \in \pi_p^{-1}(U_p)$, and hence, there is $n \in \mathbb{N}$ such that $B_n \subset \pi_p^{-1}(U_p)$. Applying π_p on both the sides, we see that $\pi_p(B_n) \subset U_p \subsetneqq X_p$. Thus $p \in J$. Consequently, $K \subset J$. This proves that at most countably many factor spaces are nontrivial.

Conversely, suppose each X_i is first countable and all but countably many X_i are indiscrete spaces. Let $K = \{i \in I : X_i \text{ is not indiscrete}\}$. Let $\mathbf{x} \in X$. For every $i \in K$, let $\mathscr{B}_i$ be a countable local base at $\mathbf{x}_i$. Let

$$\mathscr{B}_\mathbf{x} = \{\cap_{j=1}^k \pi_{i_j}^{-1}(B_{i_j}) : k \in \mathbb{N}, i_j \in K, B_{i_j} \in \mathscr{B}_{i_j}\}.$$

Clearly, $\mathscr{B}_\mathbf{x}$ is countable. For every $U \in \mathscr{U}_\mathbf{x}$, let $i_1, i_2, \ldots, i_k \in I$ be such that $\pi_{i_j}(U)$ is a proper subset of X_{i_j}. Then there are $B_{i_j} \in \mathscr{B}_{i_j}$ such that $\mathbf{x}_{i_j} \in B_{i_j} \subset U_{i_j}$. Hence $\mathbf{x} \in \cap_{j=1}^k \pi_{i_j}^{-1}(B_{i_j}) \subset U$. This proves that $\mathscr{B}_\mathbf{x}$ is a local base at $\mathbf{x}$. □

44.6. Theorem. *Let $\{X_i : i \in I\}$ be a family of topological spaces. Then $X = \prod_{i\in I} X_i$ is second countable if and only if each X_i is second countable and all but countably many X_i are indiscrete spaces.*

PROOF. Suppose that X is second countable. So, it is first countable. Hence all but countably many factor spaces are trivial. Also, since each X_i is homeomorphic to a subspace of X, and since subspace and homeomorphic image of a second countable space are second countable, it follows that each X_i is second countable.

Conversely, suppose each X_i is second countable and all but countably many X_i are indiscrete spaces. Let $K = \{i \in I : X_i \text{ is not indiscrete}\}$. For every $i \in K$, let $\mathscr{B}_i$ be a countable base at $\mathbf{x}_i$. Then

$$\mathscr{B} = \{\cap_{k=1}^n \pi_{i_k}^{-1}(B_{i_k}) : B_{i_k} \in \mathscr{B}_{i_k} \text{ for every } i_k \in K, n \in \mathbb{N}\}$$

is a countable base for the product topology. This completes the proof. □

44.7. Example. The ordered topological space $[0, \omega_1]$, where ω_1 is the first uncountable ordinal, is not first countable. Indeed, suppose $\mathscr{B}_{\omega_1} = \{B_n : n \in \mathbb{N}\}$ is a local base at ω_1. Since the basic open sets containing ω_1 are of the form $(a, \infty) = (a, \omega_1]$, $(a \in [0, \omega_1))$ we get $a_1, a_2, a_3, \cdots \in [0, \omega_1)$ such that $(a_n, \omega_1] \subset B_n$ for all $n \in \mathbb{N}$. Since $a_n \in [0, \omega_1)$, $(-\infty, a_n] = (-\infty, a_n + 1)$ is countable for the successor $a_n + 1$ of a_n. Thus $\cup_{n=1}^{\infty}(-\infty, a_n]$ is countable. Since $(-\infty, \omega_1)$ is uncountable, there is $a \in (-\infty, \omega_1) \setminus \cup_{n=1}^{\infty}(-\infty, a_n]$. Clearly, $(a, \omega_1]$ is a neighbourhood of ω_1, that does not contain any B_n. This contradiction establishes that $[0, \omega_1]$ is not first countable.

45. Separable Spaces

45.1. Definition. A topological space X is said to be *separable* if it has a countable dense set.

45.2. Example.

(1) A countable set with any topology is separable.
(2) $\mathbb{R}^n$ is separable as $\mathbb{Q}^n$ is a countable dense subset.
(3) $\mathbb{R}_\ell^n$ is separable as $\mathbb{Q}^n$ is a countable dense subset.
(4) An uncountable set with cocountable topology is not separable.
(5) A cofinite topological space is separable.
(6) A finite space is separable. Indeed the whole space is a countable dense set.
(7) A countable space is separable. Indeed the whole space is a countable dense set.

45.3. Example. A subspace of a separable space need not be separable.

Indeed $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is separable as $\mathbb{Q} \times \mathbb{Q}$ is a countable dense subset of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. However $\{(x, -x) : x \in \mathbb{R}\}$ is an uncountable discrete subspace of $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

45.4. Theorem. *Continuous image of a separable space is separable.*

Proof. Let X, Y be two topological spaces and $f : X \to Y$ be an onto continuous function. Let $A \subset X$ be dense in X. Then $Y = f(X) = f(\overline{A}) \subset \overline{f(A)} \subset Y$ shows that $\overline{f(A)}$ is dense in Y. If X is separable, we can choose A to be countable, and hence, $f(A)$ is a countable dense subset of Y. □

45.5. Theorem. *Let $\{X_i : i \in I\}$ be a family of topological spaces, and for each $i \in I$, A_i be a dense set. Then $A = \prod_{i\in I} A_i$ is dense in $X = \prod_{i\in I} X_i$.*

Proof. Let $\mathbf{x} \in X$ and $U \in \mathscr{U}_{\mathbf{x}}$. Since $U_i = \pi_i(U)$ is open, there is $\mathbf{y}_i \in U_i \cap A_i$. Clearly, $\mathbf{y} \in U \cap A$. This completes the proof. □

45.6. Corollary. *A finite product space is separable if and only if each factor space is separable.*

The following example and the Theorem following it show that an uncountable product of separable spaces may or may not be separable.

45.7. Example. Let $I \neq \emptyset$ and for each $i \in I$, $X_i = \{a, b\}$ with $\mathscr{T}_i = \{\emptyset, X_i, \{a\}\}$. Let $\mathbf{x} \in X = \prod_{i\in I} X_i$ be defined as $\mathbf{x}_i = a$, $(i \in I)$. Let $W = \pi_i^{-1}(U)$ be any subbasic neighbourhood where $U \subset X_i$ is open. Clearly, $\mathbf{x} \in W$. Thus $\{\mathbf{x}\}$ is dense in X. Thus X is separable.

45.8. Theorem. *Let $\{X_i : i \in I\}$ be a family of Hausdorff topological spaces and $X = \prod_{i\in I} X_i$ be separable. Let $K = \{i \in I : X_i$ has at least two points$\}$. Then $|K| \leq \mathfrak{c}$ and each X_i is separable.*

Proof. Since continuous image of a separable space is separable, each X_i is separable. For each $i \in K$ choose two nonempty disjoint open sets $U_i, V_i \subset X_i$. This can be achieved by taking two points in X_i and separating them by means of open sets using the fact that X_i is Hausdorff. Now let $D \subset X$ be a countable dense subset of X. For each $i \in I$, let

$$G_i = \{x \in D : \pi_i(x) \in U_i\}, \quad H_i = \{x \in D : \pi_i(x) \notin U_i\}$$

Note that for $i \neq j$, $\pi_i^{-1}(U_i) \cap \pi_j^{-1}(V_j)$ is a nonempty open set, and hence, we can choose $y \in D \cap \pi_i^{-1}(U_i) \cap \pi_j^{-1}(V_j)$. Thus $y \in G_i$ and $y \notin G_j$. Thus $i \neq j \Rightarrow G_i \neq G_j$. That is, $\{G_i : i \in I\}$ are distinct subsets of D. Thus $|K| \leq 2^{|D|} = \mathfrak{c}$. □

Since the product with a singleton does not alter the product, we can omit them. Thus essentially we proved the following.

45.9. Theorem. *Let $X = \prod_{i\in I} X_i$ be a product of Hausdorff spaces, where each X_i has at least two points. If X is separable, then $|I| \leq \mathfrak{c}$ and each X_i is separable.*

The converse is essentially due to Pondiczery [**Pon44**] and the proof presented here is due to Marczewski [**Mar47**] from Ross and Stone [**RS64**].

45.10. Theorem. *Let $X = \prod_{i\in I} X_i$ be a product of Hausdorff spaces, where each X_i has at least two points. If each X_i is separable and $|I| \leq \mathfrak{c}$, then X is separable.*

Proof. If I is finite, we are done. So we assume that I is infinite. Also, since $|I| \leq \mathfrak{c}$, we can assume that $I \subset \mathbb{R}$. Since I is infinite, we can assume that I is dense in $\mathbb{R}$. For each $i \in I$, let $D_i = \{x_n^i : n = 1, 2, 3, \dots\}$ be a countable dense subset of X_i. Define

$$P = \{r = (r_1, r_2, \dots, r_{n-1}, k_1, k_2, \dots, k_n) : r_i \in \mathbb{Q}, r_i < r_{i+1}, k_j \in \mathbb{N}, n \geq 2\}$$

For every $r \in P$, define

$$\mathbf{x}_i^r = \begin{cases} x_{k_1}^i, & \text{if } i \in (-\infty, r_1]; \\ x_{k_m}^i, & \text{if } i \in (r_{m-1}, r_m]; \\ x_{k_n}, & \text{if } i \in (r_m, \infty). \end{cases}$$

Then $\{\mathbf{x}_r : r \in P\}$ is a countable subset of X. Let $U = \cap_{j=1}^n \pi_{i_j}^{-1}(U_{i_j})$. Rearranging, if needed, we assume that $i_1 < i_2 < i_3 < \cdots < i_n$. Choose $r \in P$ such that $i_1 < r_1 < i_2 < r_2 < i_3 < r_3 < \cdots < r_{n-1} < i_n$. For $1 \leq j \leq n$, choose $x_{k_j}^{i_j} \in D_{i_j} \cap U_{i_j}$. Then $x_r \in U$. Thus $\{\mathbf{x}_r : r \in P\}$ is dense in X. □

45.11. Theorem. *A second countable space is separable.*

Proof. Let X be a topological space with base $\mathscr{B}$. Choose $x_B \in B$ for every $B \in \mathscr{B}$ and let $A = \{x_B : B \in \mathscr{B}\}$. Let $U \subset X$ be open and $x \in U$. So, there is $B \in \mathscr{B}$ such that $x \in B \subset U$. But then $x_B \in U \cap A$. Thus A is dense in X. Consequently, when X is second countable and $\mathscr{B}$ is countable, then A is countable, and hence, X is separable. □

45.12. Example. Let X be an uncountable set with the cofinite topology $\mathscr{T}_f$. Let $A \subset X$ be countably infinite. Then clearly, A is dense in X. But as noted in the Example 19.12, X is not second countable.

45.13. Theorem. *A separable metric space is second countable.*

Proof. Let (X, d) be a separable metric space. Let $A = \{x_n : n \in \mathbb{N}\}$ be a countable dense subset of X. Define

$$\mathscr{B} = \{B(x_n, r) : n \in \mathbb{N}, r > 0, r \in \mathbb{Q}\}$$

Let U be an open set and $x \in U$. Let $B(x, \varepsilon) \subset U$. Choose $x_n \in B(x, \frac{\varepsilon}{4})$ and $r \in \mathbb{Q} \cap (\frac{\varepsilon}{4}, \frac{\varepsilon}{2})$. Then for $y \in B(x_n, r) \Rightarrow d(x, y) \leq d(x, x_n) + d(x_n, y) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} < \varepsilon$. Thus $B(x_n, r) \subset B(x, \varepsilon) \subset U$. Thus $\mathscr{B}$ is a base for the topology induced by the metric. □

46. Lindelöf Spaces

46.1. Definition. Let X be a set and $Y \subset X$. A family $\mathscr{A} = \{G_i : i \in I\}$ of subsets of X is said to be a *cover for Y in X* if $Y \subset \cup\mathscr{A} = \cup_{i \in I} G_i$.

46.2. Definition. Let $(X, \mathscr{T})$ be a topological space and $Y \subset X$. Let $\mathscr{A}$ be a cover for Y in X. $\mathscr{A}$ is called

(1) a *finite cover* if I is finite.
(2) called a *countable cover* if I is countable.
(3) an *open cover* for Y in X if each $G_i \in \mathscr{A}$ is open in X, that is, if $\mathscr{A} \subset \mathscr{T}$.

$\mathscr{C} \subset \mathscr{A}$ is called a *subcover of* $\mathscr{A}$ if $\mathscr{C}$ is also a cover for Y.

46.3. Definition. Let X be a topological space. A subset Y of X is said to be *Lindelöf* in X if every cover for Y in X has a countable subcover.

The very first task at our hand is to remove the redundancy of the phrases open cover "in X" and "in Y".

46.4. Theorem. *Let X be a topological space and $Z \subset Y \subset X$. Then Z is Lindelöf in Y if and only if Z is Lindelöf in X.*

Proof. Let Z be Lindelöf in Y. Let $\mathscr{A} = \{G_i : i \in I\}$ be an open cover for Z in X. But then

$$Z \subset Z \cap Y \subset (\cup_{i\in I} G_i) \cap Y = \cup_{i\in I}(G_i \cap Y). \tag{46.4.1}$$

Since $G_i \cap Y$ is open in Y for each $i \in I$, we conclude that $\mathscr{A}_Y = \{G_i \cap Y : i \in I\}$ is an open cover for Z in Y. Since Z is Lindelöf in Y, there is a countable subcover $\mathscr{C} = \{G_{i_k} \cap Y : k \in \mathbb{N}\}$ of $\mathscr{A}_Y$. Consequently, $\mathscr{C}_X = \{G_{i_k} : k \in \mathbb{N}\}$ is a countable subcover of $\mathscr{A}$ in X.

Conversely, suppose Z is Lindelöf in X. Let $\mathscr{A} = \{U_i : i \in I\}$ be an open cover for Z in X. But then for each i, there is an open set G_i in X such that $U_i = G_i \cap Y$. Hence, $Z \subset \cup_{i\in I}(G_i \cap Y) \subset \cup_{i\in I} G_i$. Thus $\mathscr{A}_X = \{G_i : i \in I\}$ is an open cover for Z in X. But the Lindelöfness of Z in X gives a countable subcover $\mathscr{C} = \{G_{i_k} : k \in \mathbb{N}\}$ of $\mathscr{A}_X$ in X. Consequently, as in (46.4.1), $\mathscr{C}_Y = \{U_{i_k} : k \in \mathbb{N}\}$ is a countable subcover of $\mathscr{A}$ in Y. □

46.5. Example.

(1) An indiscrete space is Lindelöf.
(2) A discrete space is Lindelöf if and only if it is countable.
(3) If a topology or the set in a topological space is countable, then it is Lindelöf.

46.6. Example. A cofinite space is Lindelöf. Indeed let X be a cofinite space and $x \in X$. Let $\mathscr{A}$ be an open cover for X. Then there is $G \in \mathscr{A}$ such that $x \in G$. Since X carries the cofinite topology, $X \setminus G = \{x_1, x_2, \ldots, x_n\}$, a finite set. But then we can choose $G_1, G_2, \ldots, G_n \in \mathscr{T}_f$ such that $x_i \in G_i$ for every $1 \leq i \leq n$. Thus $X \subset G \cup \cup_{k=1}^{n} G_k$. Thus X is Lindelöf.

46.7. Theorem. *A second countable space is Lindelöf.*

Proof. Let $(X, \mathscr{T})$ be a second countable space with countable basis $\mathscr{B} = \{B_n : n \in \mathbb{N}\}$. Let $\mathscr{A} = \{G_i : i \in I\}$ be an open cover for X. For $n \in \mathbb{N}$, define

$\mathscr{A}_n = \{i \in I : B_n \subset G_i\}$. Let $J = \{n \in \mathbb{N} : \mathscr{A}_n \neq \emptyset\}$. For every $n \in J$, choose $i_n \in \mathscr{A}_n$ and let $\mathscr{C} = \{G_{i_n} : n \in J\}$. Now let $x \in X$. Then there is $j \in I$ such that $x \in G_j$. But then there is $n \in \mathbb{N}$ such that $x \in B_n \subset G_j$. But then $x \in B_n \subset G_{i_n}$. Thus $\cup\mathscr{C} = X$. This completes the proof. □

46.8. Example. A Lindelöf space need not be second countable. Indeed, let X be an uncountable set with the cocountable topology $\mathscr{T}_c$. Let $\mathscr{A} = \{G_i : i \in I\}$ be an open cover for X. Choose a nonempty $G_{i_0} \in \mathscr{A}$. Then $X \setminus G_{i_0} = \{x_n : n \in \mathbb{N}\}$, a countable set. For each $n \in \mathbb{N}$, choose $G_{i_n} \in \mathscr{A}$ such that $x_n \in G_{i_n}$. Then $X = G_{i_0} \cup \{x_n : n \in \mathbb{N}\} \subset \cup_{n=0}^{\infty} G_{i_n}$. Thus X is Lindelöf. Suppose X is second countable. So, X separable. Let $E \subset X$ be a countable dense subset. But in a cocountable space every countable set is closed. Thus $X = \overline{E} = E$, a contradiction, because X is uncountable.

46.9. Theorem. *A closed subspace of a Lindelöf space is Lindelöf.*

Proof. Let $(X, \mathscr{T})$ be a Lindelöf space and $Y \subset X$ be closed. Let $\mathscr{A} = \{G_i : i \in I\}$ be an open cover for Y. Let $\mathscr{A}' = \{U \subset X : U = G_i$ for some $i \in I$ or $U = X \setminus Y\}$. Clearly, $X = Y \cup X \setminus Y \subset \cup\mathscr{A} \cup X \setminus Y = \cup\mathscr{A}'$. Thus $\mathscr{A}'$ is an open cover for X. Let $\mathscr{C}$ be a countable subcover of $\mathscr{A}$. That is, $\mathscr{C}$ is countable and $X \subset \mathscr{C}$. Removing $X \setminus Y$ from $\mathscr{C}$ if it is there, we see that the resulting family is a countable subcover of $\mathscr{A}$. This completes the proof. □

46.10. Proposition. *Let X be a topological space and $X = \cup_{n=1}^{\infty} A_n$. If each A_n is Lindelöf, then so is X.*

Proof. Let $\mathscr{A}$ be an open cover for X. So, $\mathscr{A}$ is an open cover for each A_n. Consequently, it will have a countable subcover $\mathscr{A}_n$ for each of A_n. Thus $\cup_{n=1}^{\infty}\mathscr{A}_n$ is a countable subcover of $\mathscr{A}$ for X □

46.11. Theorem. $[0,1]$ *with the lower limit topology is Lindelöf.*

Proof. Let $\mathscr{A}$ be an open cover for $[0,1]$ in $\mathbb{R}_\ell$. Let $A = \{s \in (0,1] : [0,s]$ can be covered by countably many members of $\mathscr{A}\}$. We show that $1 \in A$.

First observe that $A \neq \emptyset$. For that, let $G \in \mathscr{A}$ such that $0 \in G$. But G is open. So, there exists an interval $[a,b)$ such that $0 \in [a,b) \subset G$. If $1 < b$, then $[0,1] \subset [a,b) \subset G$, and so, $1 \in A$. If $b \leq 1$, then $[0, \frac{b}{2}] \subset [a,b) \subset G$. Thus $\frac{b}{2} \in A$. Hence $A \neq \emptyset$. Thus $[0,r)$ is covered by countably many members of $\mathscr{A}$. Thus $r \in A$.

Now let $r = \sup A$. Clearly, $0 < r \leq 1$. We claim that $r \in A$. Let $0 < a < r$. So, a is not an upper bound of A. Hence, there is $b \in A$ such that $a < b$. But then $[0,a) \subset [0,b)$, which is covered by countably many members of $\mathscr{A}$. Let $G \in \mathscr{A}$ be such that $r \in G$. Hence, $[r, r+\varepsilon) \subset G$. If $r < 1$, then $[0, r + \frac{\varepsilon}{2})$ is covered by countably many members of $\mathscr{A}$ and hence r is not the supremum of

A, a contradiction. Thus $r = 1$. Hence $[0, 1]$ can be covered by a countably many members of A. Thus $[0, 1]$ is Lindelöf with the lower limit topology. □

46.12. Corollary. *For $-\infty < a < b < \infty$, $[a, b]$ is Lindelöf with the lower limit topology.*

Proof. Indeed $[a, b] \cong [0, 1]$, where both carry the lower limit topology. So, the result follows. □

46.13. Corollary. *$\mathbb{R}_\ell$ is Lindelöf.*

Proof. Indeed, the result follows by expressing $\mathbb{R} = \cup_{n\in\mathbb{Z}}[n, n + 1]$ and applying the Proposition 46.10 and the Corollary 46.12. □

46.14. Corollary. *$[a, b)$ with the lower limit topology is Lindelöf.*

Proof. Since a closed subspace of a Lindelöf space is Lindelöf the result follows. □

46.15. Theorem. *A Lindelöf T_3-space is T_4.*

Proof. Let $(X, \mathscr{T})$ be a Lindelöf T_3 space. Let E, F be disjoint closed subsets of X. For every $x \in E$, we note that $x \in X \setminus F$. Therefore, there is an open set U_x such that $x \in U_x \subset \overline{U_x} \subset X \setminus F$. By Lindelöfness of E, there are $U_1, U_2, \ldots$ such that $E \subset \cup_{n=1}^{\infty} U_n$, and

$$U_n \subset \overline{U_n} \subset X \setminus F, \text{ that is, for every } n,\ F \subset X \setminus \overline{U_n}. \tag{46.15.1}$$

The similar argument applied to each F, we get $V_1, V_2, \ldots$ such that $F \subset \cup_{n=1}^{\infty} V_n$, and

$$V_n \subset \overline{V_n} \subset X \setminus E, \text{ that is, for every } n,\ E \subset X \setminus \overline{V_n}. \tag{46.15.2}$$

Let

$$G_n = U_n \cap (X \setminus \overline{V_1}) \cap (X \setminus \overline{V_2}) \cap \cdots \cap (X \setminus \overline{V_n})$$

and

$$H_n = V_n \cap (X \setminus \overline{U_1}) \cap (X \setminus \overline{U_2}) \cap \cdots \cap (X \setminus \overline{U_n}).$$

It is clear from (46.15.1) and (46.15.2) that $E \subset \cup_{n=1}^{\infty} G_n$ and $F \subset \cup_{n=1}^{\infty} H_n$. Also for any $n \geq m$,

$$\begin{aligned} G_n \cap H_m &= U_n \cap [\cap_{k=1}^{n}(X \setminus \overline{V_k})] \cap V_m \cap [\cap_{k=1}^{m}(X \setminus \overline{U_k})] \\ &\subset U_n \cap V_m \cap (X \setminus \overline{V_m}) = U_n \cap \emptyset = \emptyset. \end{aligned}$$

Similarly, for $m \geq n$ also $G_n \cap H_m = \emptyset$. Thus $(\cup_{n=1}^{\infty} G_n) \cap (\cup_{n=1}^{\infty} H_n) = \emptyset$. Thus X is normal. □

46.16. Corollary. *A Lindelöf regular space is normal.*

Since every second countable space is Lindelöf, the following corollaries are at once.

46.17. Corollary. *A second countable T_3-space is T_4.*

46.18. Corollary. *A second countable regular space is normal.*

46.19. Example. A separable topological space need not be Lindelöf. Indeed $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is separable because $\mathbb{Q} \times \mathbb{Q}$ is a countable dense subset of $\mathbb{R}_\ell$. On the other hand, if it is Lindelöf, then also being regular, it would be normal by the Lemma 42.14. But as we know, it is not normal.

46.20. Theorem. *A Lindelöf metric space is second countable.*

Proof. Let (X, d) be a Lindelöf metric space. Since X is Lindelöf, for every $n \in \mathbb{N}$, the open cover $\mathscr{A}_n = \{B(x, \frac{1}{n}) : x \in X\}$ for X has a countable subcover, say $\mathscr{B}_n = \{B(x_j^n, \frac{1}{n}) : j \in \mathbb{N}\}$. Let

$$\mathscr{B} = \{B(x_j^n, \tfrac{1}{n}) : j, n \in \mathbb{N}\}.$$

Clearly, $\mathscr{B}$ is countable. Let $G \subset X$ be open and $x \in G$. Choose $n \in \mathbb{N}$ such that $B(x, \frac{1}{n}) \subset G$. Choose $B(x_j^{4n}, \frac{1}{4n})$ such that $x \in B(x_j^{4n}, \frac{1}{4n})$. Then

$$y \in B(x_j^{4n}, \tfrac{1}{4n}) \Rightarrow d(x, y) \leq d(x, x_j^{4n}) + d(x_j^{4n}, y) \leq \frac{1}{4n} + \frac{1}{4n} < \frac{1}{n}.$$

Thus $x \in B(x_j^{4n}, \frac{1}{4n}) \subset B(x, \frac{1}{n}) \subset G$. Thus X is second countable. □

Combining the results so far, we summarize the following.

46.21. Corollary. *Let (X, d) be a metric space. The following are equivalent.*

(1) X is Lindelöf.
(2) X is second countable.
(3) X is separable.

Proof. ☞(1)⇒(2) follows from the Theorem 46.20.

☞(2)⇒(3) follows from the Theorem 45.11.

☞(3)⇒(2) follows from the Theorem 45.13.

☞(2)⇒(1) follows from the Theorem 46.7. □

47. Compact Spaces

47.1. Definition. Let X be a topological space. A subset Y of X is said to be *compact* in X if every cover for Y in X has a finite subcover.

As in the case of Lindelöf spaces, here too the very first task at hand is to remove the redundancy of the phrases open cover "in X" and "in Y".

47.2. Theorem. *Let X be a topological space and $Z \subset Y \subset X$. Then Z is compact in Y if and only if Z is compact in X.*

Proof. Let Z be compact in Y. Let $\mathscr{A} = \{G_i : i \in I\}$ be an open cover for Z in X. But then

$$Z \subset Z \cap Y \subset (\cup_{i\in I} G_i) \cap Y = \cup_{i\in I}(G_i \cap Y). \tag{47.2.1}$$

Since $G_i \cap Y$ is open in Y for each $i \in I$, we conclude that $\mathscr{A}_Y = \{G_i \cap Y : i \in I\}$ is an open cover for Z in Y. Since Z is compact in Y, there is a finite subcover $\mathscr{C} = \{G_{i_k} \cap Y : 1 \leq k \leq n\}$ of $\mathscr{A}_Y$. Consequently, $\mathscr{C}_X = \{G_{i_k} : 1 \leq k \leq n\}$ is a finite subcover of $\mathscr{A}$.

Conversely, suppose Z is compact in X. Let $\mathscr{A} = \{U_i : i \in I\}$ be an open cover for Z in X. But then for each i, there will be an open set G_i in X such that $U_i = G_i \cap Y$. Hence

$$Z \subset \cup_{i\in I}(G_i \cap Y) \subset \cup_{i\in I} G_i.$$

Thus $\mathscr{A}_X = \{G_i : i \in I\}$ is an open cover for Z in X. But the compactness of Z is X gives a finite subcover $\mathscr{C} = \{G_{i_k} : 1 \leq k \leq n\}$ of $\mathscr{A}_X$ in X. Consequently, as in (47.2.1), $\mathscr{C}_Y = \{U_{i_k} : 1 \leq k \leq n\}$ is a finite subcover for Z in Y. □

47.3. Example.

(1) An indiscrete space is compact.
(2) A discrete space is compact if and only if it is finite.
(3) If a topology or the set in a topological space is finite, then it is compact.
(4) $\mathbb{R}_\ell$ is not compact because $\{[n, n+1)Ln \in \mathbb{Z}\}$ is an open cover for $\mathbb{R}_\ell$ that does not have finite subcover.

47.4. Example. A cofinite space is compact. Indeed let X be a cofinite space and $x \in X$. Let $\mathscr{A}$ be an open cover for X. Then there is $G \in \mathscr{A}$ such that $x \in G$. Since X carries the cofinite topology, $X \setminus G = \{x_1, x_2, \ldots, x_n\}$, a finite set. But then we can choose $G_1, G_2, \ldots, G_n \in \mathscr{T}_f$ such that $x_k \in G_k$ for every $1 \leq k \leq n$. Thus $X \subset G \cup \cup_{k=1}^{n} G_k$. Thus X is compact.

47.5. Theorem. *A closed subspace of a compact space is compact.*

Proof. Let $(X, \mathscr{T})$ be a compact space and $Y \subset X$ be closed. Let $\mathscr{A} = \{G_i : i \in I\}$ be an open cover for Y. Let $\mathscr{A}' = \{U \subset X : U = G_i \text{ for some } i \in I \text{ or } U = X \setminus Y\}$. Clearly, $X = Y \cup X \setminus Y = \cup\mathscr{A} \cup X \setminus Y = \cup\mathscr{A}'$. Thus $\mathscr{A}'$ is an open cover for X. Let $\mathscr{C}$ be a finite subcover of $\mathscr{A}$. That is $\mathscr{C}$ is finite and $X \subset \mathscr{C}$. Removing $X \setminus Y$ from $\mathscr{C}$, if it is there, we see that the resulting family is a finite subcover of $\mathscr{A}$. This completes the proof. □

47.6. Theorem. *A compact subset of a T_2-space is closed.*

PROOF. Let X be a T_2-space. Let $F \subset X$ be compact and $x \in X \setminus F$. For every $y \in F$, using the Hausdorffness of X, we get two disjoint open sets U_y, V_y such that $x \in U_y$, $y \in V_y$. F being a compact and $\{V_y : y \in F\}$ being an open cover for F, there are $y_1, y_2, \ldots, y_n \in F$ such that $F \subset \cup_{k=1}^n V_{y_k}$. Let $U = \cap_{k=1}^n U_{y_k}$ and $V = \cup_{k=1}^n V_{y_k}$. Then U, V are open sets, $x \in U$ and $F \subset V$. Suppose $z \in U \cap V$. Then $z \in V_{y_k}$ for some k. But $z \in U \Rightarrow z \in U_{y_k}$ for every k. Thus $z \in V_{y_k} \cap U_{y_k}$, a contradiction. Thus $U \cap V = \emptyset$. □

47.7. Corollary. *A compact T_2-space is regular.*

PROOF. Let X be a compact T_2-space. Since every T_2-space is T_1, we need only to prove that X is T_3. Let $F \subset X$ be closed and $x \in X \setminus F$. Then by the Theorem 47.6, there are two disjoint open sets U, V in X such that $x \in U$ and $F \subset V$. Thus X is T_3. Since every T_2, space is T_1, so is X. Hence X is regular. □

47.8. Example. A Lindelöf subset of a T_2-space need not be closed.

indeed recall that $\mathbb{R}_\ell$ is Lindelöf and $[0, 1)$ is a Lindelöf subset of $\mathbb{R}_\ell$. Hence, being a continuous image of a Lindelöf space, $[0, 1)$ is a Lindelöf subset of $\mathbb{R}$ with the usual topology. But it is not closed.

47.9. Theorem. *A compact T_2-space is normal.*

PROOF. Let X be a compact T_2-space. Then X is regular. We need only to prove that X is T_4. Let $E, F \subset X$ be disjoint closed subsets of X. For every $y \in F$, using the regularity of X, we get two disjoint open sets U_y, V_y such that $E \subset U_y$, $y \in V_y$. Being a closed subset of a compact space, F is compact and $\{V_y : y \in F\}$ is an open cover for F. Consequently, there are $y_1, y_2, \ldots, y_n \in F$ such that $F \subset \cup_{k=1}^n V_{y_k}$. Let $U = \cap_{k=1}^n U_{y_k}$ and $V = \cup_{k=1}^n V_{y_k}$. Then U, V are open sets, $E \subset U$ and $F \subset V$. Suppose $z \in U \cap V$. Then $z \in V_{y_k}$ for some k. But $z \in U \Rightarrow z \in U_{y_k}$ for every k. Thus $z \in V_{y_k} \cap U_{y_k}$, a contradiction. Thus $U \cap V = \emptyset$, and hence, X is regular. □

47.10. Theorem. *A finite union of compact sets is compact.*

PROOF. Let X be a topological space. It is sufficient to prove that union of two compact subsets of X is compact and the general case follows by induction. Let $A, B \subset X$ be compact. Let $\mathscr{A} = \{G_i : i \in I\}$ be an open cover for $A \cup B$. So $\mathscr{A}$ is an open cover for A as well as for B. So, there are $G_{i_1}, G_{i_2}, G_{i_3}, \ldots, G_{i_n}$ such that for some $1 \leq k \leq n$ $A \subset \cup_{j=1}^k G_{i_j}$ and $B \subset \cup_{j=k+1}^n G_{i_j}$. Thus $A \cup B \subset \cup_{j=1}^n G_{i_j}$ and so, $\{G_{i_j} : 1 \leq j \leq n\}$ is a finite subcover of $\mathscr{A}$ for $A \cup B$. □

47.11. Lemma. *Let X be a Hausdorff space and $\{E_i : i \in I\}$ be a nonempty collection of compact subsets of X. Then $\cap E_i$ is compact.*

PROOF. Fix E_j. Each E_i, being a compact subset of a Hausdorff space, is closed. Hence by the Theorem 47.6, $\cap E_i$ is a closed subset of E_j. So, $\cap E_i$ is compact by the Theorem 47.5. □

The following example shows that intersection of compact subsets need not be compact in non-Hausdorff spaces. That is, the condition that Y is Hausdorff cannot be dropped from the Lemma 47.11

47.12. Example. Let $\mathscr{T}$ be the topology on $\mathbb{N}$ generated by the base $\mathscr{B}_n = \{\{n\} : n \in \mathbb{N}, n \geq 3\} \cup \{\mathbb{N} \setminus \{1\}, \mathbb{N} \setminus \{2\}\}$. Then $\mathbb{N}$ is compact. Indeed Given any cover $\mathscr{A}$ for $\mathbb{N}$, pick $G_1, G_2 \in \mathscr{A}$ such that $1 \in G_1, 2 \in G_2$. Hence $\mathbb{N} \subset G_1 \cup G_2$. Let us also note that $\mathbb{N} \setminus \{1\}, \mathbb{N} \setminus \{2\}$ are compact subsets of $\mathbb{N}$ but their intersection, $\mathbb{N} \setminus \{1, 2\}$, is not compact.

The following is an in-house application of the Theorem 47.6.

47.13. Theorem. *Let $(X, \mathscr{T})$ be a Hausdorff space and*

$$\mathscr{T}_{cpt} = \{G \subset X : G = \emptyset \text{ or } X \setminus G \text{ is } \mathscr{T}\text{-compact}\}.$$

Then $\mathscr{T}_{cpt}$ is a topology on X, $\mathscr{T}_{cpt} \leq \mathscr{T}$, $(X, \mathscr{T}_{cpt})$ is compact and $\mathscr{T} = \mathscr{T}_{cpt}$ if and only if $(X, \mathscr{T})$ is compact.

PROOF. Clearly, $\emptyset$ and X are in $\mathscr{T}_{\text{cpt}}$. Let $\{G_i : i \in I\} \subset \mathscr{T}_{\text{cpt}}$. Let $G_j \neq \emptyset$. So, $X \setminus G_j$ is $\mathscr{T}$-compact. Note that for every $G \in \mathscr{T}_{\text{cpt}}$, either $X \setminus G = X$ or $X \setminus G$ is a $\mathscr{T}$-compact subset of X. Since $(X, \mathscr{T})$ is T_2, its compact subset must be $\mathscr{T}$-closed. Hence $X \setminus G$ must be $\mathscr{T}$-closed. Fix $j \in I$ such that $X \setminus G_j$ is $\mathscr{T}$-compact. Then $X \setminus \cup_{i \in I} G_i = \cap (X \setminus G_i)$ is a $\mathscr{T}$-closed subset of the $\mathscr{T}$compact set $X \setminus G_j$. Therefore, $X \setminus \cup_{i \in I} G_i$ is $\mathscr{T}$-compact. Thus $\cup G_i \in \mathscr{T}_{\text{cpt}}$. Also, let $G, H \in \mathscr{T}$. If $G \cup H = \emptyset$, we are done. Otherwise $X \setminus G, X \setminus H$ are $\mathscr{T}$-compact. Hence $X \setminus (G \cap H) = (X \setminus G) \cup (X \setminus H)$ is compact. Hence $G \cap H \in \mathscr{T}_{\text{cpt}}$. Thus $\mathscr{T}_{\text{cpt}}$ is a topology.

Let $\emptyset \neq G \in \mathscr{T}_{\text{cpt}}$. Then $X \setminus G$ is $\mathscr{T}$-compact. Since $(X, \mathscr{T})$ is Hausdorff, $X \setminus G$ is $\mathscr{T}$-closed. So, $G \in \mathscr{T}$. Now let $\mathscr{A}$ be an $\mathscr{T}_{\text{cpt}}$-open cover for X. Let $\emptyset \neq G \in \mathscr{A}$. So, $\mathscr{A}' = \mathscr{A} \setminus \{G\}$ is a $\mathscr{T}_{\text{cpt}}$-open cover for $X \setminus G$. Since $\mathscr{T}_{\text{cpt}} \leq \mathscr{T}$, $\mathscr{A}'$ is a $\mathscr{T}$-open cover for $X \setminus G$, which is $\mathscr{T}$-compact. Thus there are $G_1, G_2, \ldots, G_n \in \mathscr{A}' \subset \mathscr{A}$ such that $X \setminus G \subset \cup_{i=1}^{n} G_i$. Thus $\{G, G_1, G_2, \ldots, G_n\}$ is a finite $\mathscr{T}_{\text{cpt}}$-open subcover of $\mathscr{A}$ for X. Thus $(X, \mathscr{T}_{\text{cpt}})$ is compact.

If $\mathscr{T} = \mathscr{T}_{\text{cpt}}$, then by the compactness of $(X, \mathscr{T}_{\text{cpt}})$, $(X, \mathscr{T})$ is compact. Now assume that $(X, \mathscr{T})$ is compact. Let $G \in \mathscr{T}$. Then $X \setminus G$ is a $\mathscr{T}$-closed subset of the compact space $(X, \mathscr{T})$, and hence, $X \setminus G$ is $\mathscr{T}$-compact. Thus $G \in \mathscr{T}_{\text{cpt}}$. This completes the proof. □

47.14. Theorem. *A continuous image of a compact space is compact.*

PROOF. Let X be a compact space and Y be a topological space. Let $f\colon X \to Y$ be an onto continuous function. Let $\mathscr{A} = \{U_i : i \in I\}$ be an open cover for Y. Hence,

$$X = f^{-1}(Y) = f^{-1}(\cup_{i\in I} U_i) = \cup_{i\in I} f^{-1}(U_i).$$

Since f is continuous, $\{f^{-1}(U_i) : i \in I\}$ is an open cover for X. Since X is compact, there are $i_1, i_2, \ldots, i_n \in I$ such that $X = \cup_{k=1}^{n} f^{-1}(U_{i_k})$. But then

$$Y = f(X) = f(\cup_{k=1}^{n} f^{-1}(U_{i_k})) = \cup_{k=1}^{n} f(f^{-1}(U_{i_k})) \subset \cup_{k=1}^{n} U_{i_k}. \qquad \square$$

47.15. Corollary. *Let $\mathscr{T}_1, \mathscr{T}_2$ be two topologies on a set X such that $\mathscr{T}_1 \leq \mathscr{T}_2$. If $(X, \mathscr{T}_2)$ is compact, then so is $(X, \mathscr{T}_1)$.*

PROOF. Indeed, $(X, \mathscr{T}_1)$ is a continuous image of $(X, \mathscr{T}_2)$ under the identity map $\iota_X\colon (X, \mathscr{T}_2) \to (X, \mathscr{T}_1)$. Hence the result follows. $\square$

47.16. Corollary. *Homeomorphic image of a compact space is compact.*

The following is an automatic continuity result of the inverse of a bijective continuous map.

47.17. Theorem. *A bijective continuous map from a compact space to a Hausdorff space is a homeomorphism.*

PROOF. Let X be a compact space, Y be a Hausdorff space and $f\colon X \to Y$ be a bijective continuous map. Let $F \subset X$ be closed. Then it is compact. But then by the Theorem 47.14, $f(F)$ is compact. Since Y is Hausdorff, by the Theorem 47.6, $f(F)$ is closed. Thus f is a closed map, and hence, a homeomorphism. $\square$

47.18. Example. In the square $X = [0,1] \times [0,1]$ define $(0,t) \sim (1,t)$ for $(0 \leq t \leq 1)$. Then the resulting quotient space $X/\!\sim$ is homeomorphic to the cylinder $S^1 \times [0,1]$. Indeed define $f\colon X \to S^1 \times [0,1]$ by $f(s,t) = ((\cos s, \sin s), t) = (e^{is}, t)$, $(s,t) \in X$. It is clear that $(s,t) \in X \mapsto (\cos s, \sin s) \in S^1$ and $(s,t) \in X \mapsto t \in [0,1]$ are continuous and hence by the Theorem 22.9, f is continuous. But then since X is compact and $S^1 \times [0,1]$ is Hausdorff, f is a closed map. Hence by the Theorem 21.26(2), f is a quotient map. Thus $X/\{\sim\}$ is homeomorphic to $S_1 \times [0,1]$

47.19. Definition. Let X be a set and $\mathscr{F}$ be a collection of subsets of X. We say that $\mathscr{F}$ has the *Finite Intersection Property (FIP)* if every finite subfamily of $\mathscr{F}$ has nonempty intersection. That is, if $A_1, A_2, \ldots, A_n \in \mathscr{F}$, then $\cap_{i=1}^{n} A_i \neq \emptyset$. In this case, we also say that $\mathscr{F}$ is with FIP.

47.20. Theorem. *Let X be a topological space. Then the following are equivalent.*

(1) X is compact.

(2) Every family of closed subsets of X with FIP has nonempty intersection. That is, for every family $\mathscr{F}$ of closed subsets of X, if $\mathscr{F}$ has FIP, then $\cap\mathscr{F} \neq \emptyset$.

PROOF. ☞(1) ⇒ (2) Suppose X is compact and $\mathscr{F}$ is a family of closed subsets of X with FIP. Suppose, if possible, that $\cap\mathscr{F} = \emptyset$. Define

$$\mathscr{A} = \{X \setminus E : E \in \mathscr{F}\}. \tag{47.20.1}$$

Then $\mathscr{A}$ is a collection of open subsets of X. Also,

$$\cup\mathscr{A} = \cup_{E\in\mathscr{F}}(X \setminus E) = X \setminus \cap_{E\in\mathscr{F}}E = X \setminus \cap\mathscr{F} = X \setminus \emptyset.$$

Thus $\mathscr{A}$ is an open cover for X. Since X is compact, there are $E_1, E_2, \ldots, E_k \in \mathscr{F}$ such that $\cup_{i=1}^{k}(X \setminus E_i) = X$. Hence

$$\cap_{i=1}^{k}E_i = X \setminus (X \setminus \cap_{i=1}^{k}E_i) = X \setminus \cup_{i=1}^{k}(X \setminus E_i) = X \setminus X = \emptyset,$$

a contradiction as $\mathscr{F}$ has FIP. Thus our supposition is wrong. Thus $\cap\mathscr{F} \neq \emptyset$.

☞(2) ⇒ (1) Suppose every family of closed subsets of X with FIP has nonempty intersection. Let $\mathscr{A}$ be an open cover for X, Define

$$\mathscr{F} = \{X \setminus E : E \in \mathscr{A}\}.$$

Then $\mathscr{F}$ is a collection of closed subsets of X. Also,

$$\cap\mathscr{F} = \cup_{E\in\mathscr{A}}(X \setminus E) = X \setminus \cup_{E\in\mathscr{F}}E = X \setminus \cup\mathscr{A} = X \setminus X = \emptyset.$$

Thus intersection of whole family $\mathscr{F}$ is empty. Hence, $\mathscr{F}$ cannot have FIP. So, there exist $E_1, E_2, \ldots, E_k \in \mathscr{A}$ such that $\cap_{i=1}^{k}(X \setminus E_i) = \emptyset$. Hence $\cup_{i=1}^{k}E_i = X$. Thus $\{E_1, E_2, \ldots, E_k\}$ is a finite subcover of $\mathscr{A}$. Hence X is compact. □

47.21. Theorem. *A compact subset of a metric space is closed and bounded.*

PROOF. Let (X, d) be a metric space and $F \subset X$ be compact. Fix $x \in X$. Then clearly, $\mathscr{A} = \{B(x, r) : r > 0\}$ is an open cover for F. So, it has a finite subcover. Say, $\{B(x, r_k) : k = 1, 2, 3, \ldots, n\}$. Without loss of generality we can assume that $r_1 \geq r_k$ for every $k = 2, 3, \ldots, n$. But then $F \subset \cup_{k=1}^{n}B(x, r_k) \subset B(x, r_1)$. Hence F is bounded. Clearly, since a metric space is Hausdorff, F, being a compact subset of a Hausdorff space, is closed. □

47.22. Corollary. *A compact subset of $\mathbb{R}$ is closed and bounded.*

The important fact is that converse to the Theorem 47.21 is not true but the converse to the Corollary 47.22 is true and is known as the Heine-Borel Theorem.

47.23. Theorem. *$[0,1]$ is compact.*

PROOF. Let $\mathscr{A}$ be an open cover for $[0,1]$ in $\mathbb{R}$. Let $A = \{s \in (0,1] : [0,s]$ can be covered by finitely many elements of $\mathscr{A}\}$. We show that $1 \in A$.

First observe that $A \neq \emptyset$. For that, let $G \in \mathscr{A}$ such that $0 \in G$. But G is open. So, there exists an open interval (a,b) such that $0 \in (a,b) \subset G$. If $1 < b$, then $[0,1] \subset (a,b) \subset G$, and so, $1 \in A$. If $b \leq 1$, then $[0, \frac{b}{2}] \subset (a,b) \subset G$. Thus $\frac{b}{2} \in A$. Hence $A \neq \emptyset$.

Now let $r = \sup A$. Clearly, $0 < r \leq 1$. Fix $G_0 \in \mathscr{A}$ such that $r \in G_0$. Hence there are a_0, b_0 such that $r \in (a_0, b_0) \subset G_0$. Without loss of generality, we assume that $0 \leq a_0$. Suppose, if possible that $r \notin A$. Since a_0 is not an upper bound of A, there is $c \in A$ such that $a_0 < c$. Hence $[0,c]$ can be covered by finitely many members of $\mathscr{A}$. Consequently, $[0,r] \subset [0,c] \cup (a_o, b_0) \subset [0,c] \cup G_0$. Hence $[0,r]$ can be covered by finitely many members of $\mathscr{A}$. Assume that $r < 1$. Then $[0, \frac{r+b_0}{2}] \subset [0,r] \cup G_0$ can be covered by finitely many members of $\mathscr{A}$ and so r is not the supremum of A, a contradiction. Thus $r = 1$ and hence $[0,1]$ can be covered by finitely many members of $\mathscr{A}$. This completes the proof. □

47.24. Corollary. *$[a,b]$ is compact.*

PROOF. Follows from the fact that $f\colon [0,1] \to [a,b]$ defined by $f(t) = (b-a)t + a$, $(t \in [0,1])$, is a homeomorphism. □

47.25. Theorem. (Heine-Borel theorem) *A closed and bounded subset of $\mathbb{R}$ is compact.*

PROOF. Let $E \subset \mathbb{R}$ be closed and bounded. So there is $r \geq 0$ such that $E \subset [0,r]$. Hence E is closed subset of $[0,r]$. Since $[0,r]$ is compact, E is a closed subset of a compact space, and hence, is compact. □

47.26. Theorem. *A real-valued continuous function on a compact space is bounded and attains its minimum as well as maximum.*

PROOF. Let X be a compact space and $f\colon X \to \mathbb{R}$ be a continuous function. So $f(X)$ is compact, and hence, bounded. Thus f is bounded. Let, $m = \inf\{f(x) : x \in X\}$ and $M = \sup\{f(x)\colon x \in X\}$. For each $n \in \mathbb{N}$, $m + \frac{1}{n}$ is not a lower bound of $f(X)$ and $M - \frac{1}{n}$ is not an upper bound of $f(X)$. So, there are $x_n, y_n \in X$ such that $m \leq f(x_n) < m + \frac{1}{n}$ and $M - \frac{1}{n} \leq f(y_n) \leq M$. Taking $n \to \infty$, we see that $m, M \in \overline{f(X)}$. Since $f(X)$ is compact, it is closed. So, $m, M \in f(X)$. Thus there are $x, y \in X$ such that $M = f(x)$ and $M = f(y)$. □

Now we turn our attention to the compact subsets of ordered spaces.

47.27. Theorem. *A compact subset of an ordered space admits minimum and maximum.*

PROOF. Let X be an ordered space and $C \subset X$ be compact. Suppose C does not admit minimum. Let $c \in C$. Then there is $a \in C$ such that $a < c$. Hence $c \in (a, \infty)$, for some $a \in C$. That is, $\{(a, \infty) : a \in C\}$ is an open cover for C. Since C is compact, there are $a_1 > a_2 > \cdots > a_n$ in C such that $C \subset \cup\{(a_i, \infty) : a_i \in C, 1 \leq i \leq n\} \subset (a_n, \infty)$, giving a contradiction as $a_n \in C$ but $a_n \notin (a_n, \infty)$. Thus C must admit the minimum. Changing the rôle of $<$ with $>$, we also conclude that C admits the maximum. □

47.28. Example. Ω_0 is not compact. Indeed Ω_0 does not have maximum.

47.29. Corollary. *A compact subset of an ordered space is closed and bounded.*

PROOF. Let X be an ordered space and $C \subset X$ be compact. Then C admits minimum, say m, and maximum, say M. Then $C \subset [m, M]$. Also, X is T_2, and hence, C is closed. □

The converse to the above Corollary fails. Also, in general, a closed interval in an ordered space fail to be compact.

47.30. Example. The subset $[0, 1]$ of the ordered space $\mathbb{Q}$ is closed, bounded and admits minimum as well as maximum but is not compact.

47.31. Theorem. *Let X be an ordered space with the least upper bound property and $a, b \in X$. Then $[a, b]$ is compact.*

PROOF. Let $\mathscr{A}$ be an open cover for $[a, b]$ and let $C = \{x \in [a, b] : [a, x]\}$ can be covered by a finite subcover of $\mathscr{A}$. Let $a \in G \in \mathscr{A}$. Thus $a \in C$. Since $C \subset [a, b]$, b is an upper bound of C. Let $c = \sup C$. Clearly, $a \leq c \leq b$. Let $x \in C$ and $\mathscr{A}_0$ be a finite subcover of $\mathscr{A}$ such that $[a, x] \subset \cup\mathscr{A}_0$.

First we show that $(x, b] \cap C \neq \emptyset$.

Suppose y is the immediate successor of x. Let $y \in G_1 \in \mathscr{A}$. Clearly, $y \leq b$ otherwise y cannot be the successor of x. But then $\mathscr{A}_0$ together with G_1 covers $[a, x] \cup \{y\} = [a, y]$. Hence $y \in (x, b] \cap C$.

Suppose x does not have the immediate successor. Let $x \in G \in \mathscr{A}$. Then there is a basic open set $B = (e, \infty)$ or $B = (-\infty, d)$ or $B = (e, d)$ such that $x \in B \subset G$. If $B = (e, \infty)$, then $\mathscr{A}_0 \cup \{G\}$ is a finite subcover $[a, b]$. In the remaining cases $[x, d) \subset G$. Since x does not have the immediate successor, d is not the immediate successor of x. Hence there is $y \in (x, d)$. But then $[a, y]$ is covered by $\mathscr{A}_0 \cup \{G\}$. Thus $(x, b] \cap C \neq \emptyset$.

Next we show that $c \in C$.

Suppose that c has the immediate predecessor e. Then $a \leq e$. If e is an upper bound of C, then c is not the least upper bound of C. Thus e cannot be an upper

bound of C. So, there is $f \in C$ such that $e < f$. But then $e < f \leq c$ forces the immediate successor c of e to coincide with f. Thus $c = f \in C$.

Suppose that c does not have predecessor. Let $c \in G \in \mathscr{A}$. Then similar to the argument given for x, there will be an open set G and d such that $c \in (d, c] \subset G$. Since there is no predecessor to c, there is $y \in (d, c)$. Since $y < c$, it is not an upper bound of C. Hence there is $z \in C$ such that $y < z$. Thus $d < y < z \leq c$. Hence $[a, c] = [a, z] \cup [z, c]$, which is covered by $\mathscr{A} \cup G$. Thus $c \in C$.

Finally, suppose that $c \neq b$. Now as shown earlier $(c, b] \cap C \neq \emptyset$. Let $z \in (c, b] \cap C$. But then $c < z$, and hence, c is not an upper bound of C, a contradiction. Thus $c = b$. This completes the proof. □

47.32. Corollary. *For a well-ordered set X and $a, b \in X$, $[a, b]$ is compact.*

47.33. Corollary. *For any ordinal α, $[0, \alpha]$ is compact.*

47.34. Definition. A topological space X is said to be *sequentially compact* if every sequence has a convergent subsequence.

47.35. Example. A finite topological space is sequentially compact. Indeed, a sequence in a finite space must have an infinitely recurring term, and hence the sequence has a constant subsequence, which converges.

47.36. Definition. A topological space X is said to be a *Bolzano-Weierstrass space* if every infinite subset of X has a limit point.

The following theorem shows that every compact space is a Bolzano-Weierstrass space.

47.37. Theorem. (Bolzano-Weierstrass Property) *An infinite subset of a compact space has a limit point.*

Proof. Let X be a compact space and $A \subset X$ be infinite. Suppose, if possible, that $A' = \emptyset$. So that $\overline{A} = A \cup A' = A \cup \emptyset = A$. Thus A is closed. Also let $a \in A$. Since $a \notin A'$, there is an open set G containing a such that $G \cap A = \{a\}$. This shows that each singleton in A is open in A. That is, A is discrete. But a discrete compact space cannot be infinite. Thus A reduces to a finite set, a contradiction. Thus A must have a limit point. □

47.38. Theorem. *A sequentially compact space is a Bolzano-Weierstrass space.*

PROOF. Let X be a sequentially compact space. Let $A \subset X$ be infinite. Choose distinct termed sequence $\{x_n\}$ from A. Then passing onto a subsequence if needed, we assume that $\{x_n\}$ converges to a point $x \in X$. Let $U \in \mathscr{U}_x$. Then there is $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. That means $U \cap A$ is an infinite set. Thus $x \in A'$, and hence, X is a Bolzano-Weierstrass space. □

47.39. Theorem. *Let (X, d) have the Bolzano-Weierstrass property. Then X is sequentially compact.*

PROOF. Let $\{x_n\}$ be a sequence in X. If a term is repeated infinitely many times, say $x_{n_1} = x_{n_2} = x_{n_3} = \cdots$. Then $\{x_{n_k}\}$ is a convergent subsequence. So, we assume that no term of the sequence $\{x_n\}$ repeats infinitely. Thus the range of $\{x_n\}$ is an infinite set, and hence, it has a limit point. Let x be a limit point of the set $\{x_n\}$. Choose $n_1 \in \mathbb{N}$ such that $x_{n_1} \in B(x, 1)$. Inductively, choosing $n_1 < n_2 < \cdots < n_k$ so that $x_{n_j} \in B(x, \frac{1}{j})$, for $1 \leq j \leq k$, choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in B(x, \frac{1}{k+1})$. Clearly, $x_{n_k} \to x$ as $k \to \infty$. This completes the proof. □

47.40. Theorem. *A compact metric space is complete and totally bounded.*

PROOF. Let (X, d) be a compact metric space. Let $\varepsilon > 0$. Since X is compact, the open cover $\{B(x, \varepsilon) : x \in X\}$ for X has a finite subcover $\{B(x_i, \varepsilon) : 1 \leq i \leq n\}$. Thus $X = \cup_{i=1}^{n} B(x_i, \varepsilon)$. Thus X is totally bounded.

Let $\{x_n\}$ be a Cauchy sequence in X. Since X is compact, it is sequentially compact. So that $\{x_n\}$ has a subsequence that converges in X. But then by the Theorem 26.6, the sequence $\{x_n\}$ itself converges in X. Thus (X, d) is complete. □

47.41. Definition. Let (X, d) be a metric space and $\mathscr{A}$ be an open cover for X. A real number $r > 0$ is said to be a *Lebesgue number* of $\mathscr{A}$ if for every subset $A \subset X$ with diam diam$(A) < r$, there is $G \in \mathscr{A}$ such that $A \subset G$.

47.42. Example.

(1) A Lebesgue number of $\mathscr{A} = \{\mathbb{R}\}$ is ∞. In fact, every $r > 0$ is a Lebesgue number of $\mathscr{A}$.
(2) $\mathscr{A} = \{(\frac{1}{n+2}, \frac{1}{n}) : n \in \mathbb{N}\}$ does not have any Lebesgue number.
(3) if r is a Lebesgue number of an open cover $\mathscr{A}$ of a metric space, then every $s \in (0, r)$ is a Lebesgue number of $\mathscr{A}$.

47.43. Theorem. (Lebesgue Covering Lemma) *Let (X, d) be a sequentially compact metric space and $\mathscr{A}$ be an open cover for X. Then $\mathscr{A}$ has a Lebesgue number.*

PROOF. Suppose $\mathscr{A}$ does not have a Lebesgue number. Since 1 is not a Lebesgue number of $\mathscr{A}$, there is $A_1 \subset X$ such that $\operatorname{diam}(A_1) < 1$ and A_1 is not contained in any single member of $\mathscr{A}$. After constructing $A_2, A_3, \ldots, A_n \subset X$ such that $\operatorname{diam}(A_j) < \frac{1}{j}$ and no A_j is contained in a single member of $\mathscr{A}$, we note that $\frac{1}{n+1}$ is not a Lebesgue number of $\mathscr{A}$. So, we get $A_{n+1} \subset X$ such that $\operatorname{diam}(A_{n+1}) < \frac{1}{n+1}$ and A_{n+1} is not contained in any single member of $\mathscr{A}$. For every $n \in \mathbb{N}$, choose $x_n \in A_n$. By passing to a subsequence, if needed, we assume that $x_n \to x$. We can do so because (X, d) is sequentially compact. Let $G \in \mathscr{A}$ such that $x \in G$. So there will be $r > 0$ such that $B(x, r) \subset G$. Since $x_n \to x$, we get $N \in \mathbb{N}$ such that $d(x_n, x) < \frac{r}{2}$ whenever $n \geq N$. Select $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{r}{2}$ and $k \geq N$. Note that $x_k \in B(x, \frac{r}{2}) \cap A_k$. Also, for any $y \in A_k$, $d(x, y) \leq d(x, x_k) + d(x_k, y) < \frac{r}{2} + \frac{1}{k} < \frac{r}{2} + \frac{r}{2} = r$. Thus $A_k \subset B(x, r) \subset G$. That is A_k is contained in a single member of $\mathscr{A}$, contrary to the construction of A_k. Thus $\mathscr{A}$ must have a Lebesgue number. □

47.44. Theorem. *A sequentially compact metric space is totally bounded.*

PROOF. Let (X, d) be a sequentially compact metric space. Let $\varepsilon > 0$ be given. Let $x_1 \in X$. If $X \subset B(x_1, \varepsilon)$, then we are done. Otherwise, choose $x_2 \in X \setminus B(x_1, \varepsilon)$. If $X \subset B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$, we are done. Otherwise choose $x_3 \in X \setminus B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$. If in finitely many steps, $X \subset B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \cdots \cup B(x_k, \varepsilon)$, then we are done. Otherwise we get a sequence $\{x_n\}$ in X such that $x_n \notin B(x_1, \varepsilon) \cup B(x_2, \varepsilon) \cup \cdots \cup B(x_{n-1}, \varepsilon)$. Thus $d(x_n, x_m) \geq \varepsilon$. So that $\{x_n\}$ cannot have a subsequence, which converges. This is a contradiction to the hypothesis. Thus X must be totally bounded. □

47.45. Theorem. *A sequentially compact metric space is compact.*

PROOF. Let (X, d) be a sequentially compact metric space. Let $\mathscr{A}$ be an open cover for X. As proved in the Lebesgue Covering Lemma, the Theorem 47.43, let r be a Lebesgue number of $\mathscr{A}$. Let $\varepsilon = \frac{r}{4}$. Let $\{x_1, x_2, \ldots, x_n\}$ be an ε-net. Clearly, $\operatorname{diam}(B(x_k, \varepsilon)) < 2\varepsilon < r$. Thus for every x_k, there is $G_k \in \mathscr{A}$ such that $B(x_k, \varepsilon) \subset G_k$. But then $X \subset \cup_{k=1}^{n} B(x_k, \varepsilon) \subset \cup_{k=1}^{n} G_k$. That is $\{G_1, G_2, \ldots, G_n\}$ is a finite subcover of $\mathscr{A}$. Thus X is compact. □

Combining all the results of compact metric spaces, we have the following.

47.46. Corollary. *For a metric space (X, d), the following are equivalent.*

(1) X is compact.
(2) X is a Bolzano-Weierstrass space.
(3) X is sequentially compact.

47.47. Theorem. *The following are equivalent for a topological space X.*

(1) X is compact.
(2) Every filter on X clusters.
(3) Every net in X clusters.
(4) Every ultranet in X converges.
(5) Every ultrafilter on X converges

PROOF. ☞(1) ⇒ (2) Assume (1). Let $\mathscr{F}$ be a filter on X. Since $\mathscr{F}$ is closed under finite intersection and since $\emptyset \notin \mathscr{F}$,

$$\{\overline{E} : E \in \mathscr{F}\} \tag{47.47.1}$$

is a family of closed subsets with FIP. Since, X is compact, the family (47.47.1) has nonempty intersection. Say, $x \in \cap\{\overline{E} : E \in \mathscr{F}\}$. That, is $x \in \overline{E}$ for every $E \in \mathscr{F}$. That means $U \cap E \neq \emptyset$, whenever $E \in \mathscr{F}$ and $U \in \mathscr{U}_x$. Thus $\mathscr{F}$ clusters at x.

☞(2) ⇒ (3) Assume (2). Let $\{x_\lambda\}$ be a net in X. But then the filter generated by $\{x_\lambda\}$ clusters. Hence $\{x_\lambda\}$ clusters.

☞(3) ⇒ (4) Assume (3). Let $\{x_\lambda\}$ be an ultranet in X. Since it clusters, it converges because it is an ultranet.

☞(4) ⇒ (5) Assume (4). Let $\mathscr{F}$ be an ultrafilter on X. So, the net based on $\mathscr{F}$ is an ultranet, and hence, that net converges. So, $\mathscr{F}$ converges.

☞(5) ⇒ (1) Let $\mathscr{A}$ be an open cover for X. Define

$$\mathscr{C} = \{X \setminus \cup_{k=1}^{n} G_k : n \in \mathbb{N} \text{ and } G_k \in \mathscr{A} \text{ for all } 1 \leq k \leq n\}. \tag{47.47.2}$$

Suppose, if possible, that $\emptyset \notin \mathscr{C}$. Let $E = X \setminus \cup_{k=1}^{n} G_k, F = X \setminus \cup_{k=n+1}^{m} G_k \in \mathscr{F}$. Then $E \cap F = X \setminus \cup_{k=1}^{m} G_k$. Thus $\mathscr{C}$ is a filter base. Let $\mathscr{F}$ be a filter generated by $\mathscr{C}$ and $\mathscr{G}$ be an ultrafilter containing $\mathscr{F}$. But then by the assumption, $\mathscr{G}$ converges to some point, say x in X. Since $\mathscr{A}$ is an open cover for X, $x \in G$ for some $G \in \mathscr{A}$. That is $G \in \mathscr{U}_x \subset \mathscr{G}$. But $X \setminus G \in \mathscr{F} \subset \mathscr{G}$. Thus $G, X \setminus G \in \mathscr{G}$, a contradiction. Thus $\emptyset \in \mathscr{F}$. So, $X \setminus \cup_{k=1}^{n} G_k = \emptyset$, for some $G_1, G_2, \ldots, G_n \in \mathscr{A}$. Hence $\cup_{k=1}^{n} G_k = X$. Thus $\mathscr{A}$ has a finite subcover. □

47.48. Theorem. (Tychonoff) *A product space is compact if and only if each factor space is compact.*

PROOF. Let $\{X_i : i \in I\}$ be a family of topological spaces and $X = \prod_{i \in I} X_i$. Assume that X is compact. Since projections are continuous and onto, each X_i is a continuous image of the compact space X. Hence X_i is compact for each $i \in I$.

Conversely, suppose each X_i is compact. Let $\{\mathbf{x}_\lambda\}$ be an ultranet in X. The $\{\pi_i(\mathbf{x}_\lambda)\}$ is an ultranet in X_i for each $i \in I$. Because of the compactness of X_i, $\{\pi_i(\mathbf{x}_\lambda)\}$ converges in X_i for every $i \in I$. But then $\{\mathbf{x}_\lambda\}$ converges in X. Thus X is compact. □

47.49. Theorem. (Tube Lemma) *Let X be topological space, $x \in X$, Y be a compact space and $U \subset X \times Y$ be an open set such that $\{x\} \times Y \subset U$, then there exists an open set $V \subset X$ such that $V \times Y \subset U$.*

PROOF. For every $b \in Y$, $(x, b) \in U$, so that there is an open set $G_b \subset X, H_b \subset Y$ such that $(x, b) \in G_b \times H_b \subset U$. Clearly, $\{H_b : b \in Y\}$ is an open cover for Y, and hence, there are $b_1, b_2, \ldots, b_n \in Y$ such that $Y \subset \cup_{k=1}^{n} H_{b_i}$. Taking $V = \cap_{k=1}^{n} G_{b_i}$, we see that $V \times Y \subset U$. □

In fact a more general result holds.

47.50. Theorem. (Generalized Tube Lemma) *Let X, Y be topological spaces and $A \subset X$, $B \subset Y$ be compact. Let $U \subset X \times Y$ be an open set such that $A \times B \subset U$. Then there are open sets $G \subset X, H \subset Y$ such that $A \times B \subset G \times H \subset U$.*

PROOF. For $a \in A$, by the compactness of B, there is an open set $G_a \subset X$ containing a such that $\{a\} \times B \subset G_a \times B \subset U$. Compactness of A gives $a_1, a_2, \ldots, a_n \in A$ such that $A \subset G = \cup_{i=1}^{n} G_{a_i}$. Hence $A \times B \subset G \times B \subset U$. Thus for every $b \in B$, $A \times \{b\} \subset G \times \{b\} \subset U$. Using the compactness of A, we get an open set $H_b \subset Y$ such that $G \times \{b\} \subset G \times H_b \subset U$. Since B is compact, there are $b_1, b_2, \ldots, b_k$ such that $B \subset H = \cup_{i=1}^{k} H_{b_i}$. Thus $A \times B \subset G \times H \subset U$. □

47.51. Theorem. *Let X be topological space and Y be a compact space. Then the projection $\pi_X : X \times Y \to X$ is closed.*

PROOF. Let $F \subset X \times Y$ be closed and $x \in X \setminus \pi_X(F)$. Then $(\{x\} \times Y) \cap F = \emptyset$. That is, $\{x\} \times Y \subset (X \times Y) \setminus F$, an open set. But then by the Tube Lemma, there is an open set $V \subset X$ such that $\{x\} \times Y \subset V \times Y \subset (X \times Y) \setminus F$. That is $x \in V \subset X \setminus \pi_X(F)$. Hence, x is not a limit point of $\pi(F)$. That is, $\pi(F)$ is closed in X. □

Problems

235. Let X be a T_2-space and E, F be two disjoint compact subsets of X. Show that there are disjoint open sets $U, V \subset X$ such that $E \subset U$ and $F \subset V$.

236. Let X be a compact space and $\{F_i : i \in I\}$ be a family of closed subsets. Let U be an open set such that $\cap_{i \in I} F_i \subset U$. Show that there are $i_1, i_2, i_3, \ldots, i_n \in I$ such that $\cap_{k=1}^{n} F_{i_k} \subset U$.

237. Prove the Generalized Tube Lemma independently without using the Tube Lemma and prove the Tube Lemma as a corollary to it.

48. Countably Compact Spaces

A Lindelöf space guarantee that each of its open cover has countable subcover. In a compact space every open cover has a finite subcover. Between them is the bridge of countably compact spaces.

48.1. Definition. A topological space X is said to be *countably compact* if every countable cover for X has a finite subcover.

48.2. Remark. It is clear that a topological space is compact if and only if it is a countably compact as well as a Lindelöf space.

48.3. Example. The space $\Omega_0 = \Omega \setminus \omega_1 = [0, \omega_1)$ is countably compact but neither Lindelöf nor compact.

Indeed suppose, if possible, an open cover $\mathscr{A} = \{U_n : n \in \mathbb{N}\}$ of Ω_0 does not have a finite subcover. Choose $\alpha_n \in \Omega_0 \setminus \cup_{k=1}^{n} U_k$. Let $\alpha = \sup\{\alpha_n : n \in \mathbb{N}\}$. Then by the Theorem 6.31, $\alpha \in \Omega_0$ and by the Corollary 47.33, $[0, \alpha]$ is compact. Hence there is $n \in \mathbb{N}$ such that $[0, \alpha] \subset \cup_{k=1}^{n} U_k$. Thus $\alpha_n \in \cup_{k=1}^{n} U_k$, a contradiction. Thus Ω_0 is countably compact. We have already noted in the Example 47.28 that Ω_0 is not compact.

48.4. Example. $\mathbb{Z}$ with the discrete topology is Lindelöf but neither compact nor countably compact.

Indeed, every countable set is Lindelöf. On the other hand, a compact discrete space must be finite. Hence $\mathbb{Z}$ is not compact, and hence, not countably compact.

48.5. Theorem. *Continuous image of a countably compact space is countably compact.*

Proof. Let X, Y be topological spaces and $f : X \to Y$ be a continuous function. Suppose X is countably compact. Let $\mathscr{A} = \{U_n : n \in \mathbb{N}\}$ be an open cover for Y. Then $\{f^{-1}(U_n) : n \in \mathbb{N}\}$ is an open cover for X. Since X is countably compact, there is $k \in \mathbb{N}$ such that $X = \cup_{n=1}^{k} f^{-1}(U_n)$. But then $Y = \cup_{n=1}^{k} U_n$. This completes the proof. □

48.6. Theorem. *A real-valued continuous function on a countably compact space is bounded.*

Proof. Let X be a countably compact topological space and $f : X \to \mathbb{R}$ be a continuous function. Clearly, $X = \cup_{n=1}^{\infty} f^{-1}((-n, n))$. By the countably compactness of X, there is $k \in \mathbb{N}$ such that $X = \cup_{n=1}^{k} f^{-1}((-n, n))$. Thus $f(X) \subset (-k, k)$. That is, f is bounded. □

48.7. Theorem. *A closed subspace of a countably compact space is countably compact.*

PROOF. Let $(X, \mathscr{T})$ be a countably compact space and $Y \subset X$ be closed. Let $\mathscr{A} = \{G_n : n \in \mathbb{N}\}$ be an open cover for Y. Let $\mathscr{A}' = \mathscr{A} \cup \{X \setminus Y\}$. Clearly, $X = Y \cup X \setminus Y \subset \cup\mathscr{A} \cup X \setminus Y = \cup\mathscr{A}'$. Thus $\mathscr{A}'$ is a countable open cover for X. Let $\mathscr{C}$ be a finite subcover of $\mathscr{A}$. That is, $\mathscr{C}$ is finite and $X \subset \mathscr{C}$. Removing $X \setminus Y$ from $\mathscr{C}$, if it is there, we see that the resulting family is a finite subcover of $\mathscr{A}$. This completes the proof. □

The following characterizes a completely regular space.

48.8. Theorem. *The following are equivalent for a topological space X.*

(1) X is completely regular.
(2) X is homeomorphic to a subspace of some cube.
(3) X is homeomorphic to a subspace of some compact T_2-space.
(4) X is homeomorphic to a subspace of a normal space.

PROOF. ☞(1) ⇒ (2) We proved this in the Corollary 41.18.

☞(2) ⇒ (3) By the Tychonoff's theorem, a cube is compact T_2.

☞(3) ⇒ (4) We proved in the Theorem 47.9 that every compact T_2-space is normal.

☞(4) ⇒ (1) Follows from the Corollary 43.4. □

Problems

49. Locally Compact Spaces

Local version of the compactness is defined in two different ways. In general they are not equivalent. However, in the presence of Hausdorff condition, both of them are equivalent.

49.1. Definition. Let X be a topological space and $x \in X$. X is said to be *locally compact at x* if there is a compact neighbourhood of x. X is said to be *locally compact* if X is locally compact at each $x \in X$.

49.2. Example.

(1) Every compact space is locally compact.
(2) $\mathbb{R}$ is locally compact but not compact.
(3) A discrete space is locally compact.
(4) $\mathbb{R}_\ell$ is not locally compact.

49.3. Theorem. *Let X be a Hausdorff space. Then X is locally compact if and only if every $x \in X$ has a neighbourhood base consisting of compact sets.*

PROOF. Suppose X is locally compact. Let $x \in X$ and $\mathscr{B}_x = \{G \in \mathscr{U}_x : G \text{ is compact}\}$. Let $V = V^\circ \in \mathscr{U}_x$ and $C \in \mathscr{B}_x$. Let $U = (C \cap V)^\circ$. Then $U = U^\circ \in \mathscr{U}_x$ and $U \subset C$. Hence $\overline{U}$ is a compact neighbourhood of x. But then $\overline{U}$ is compact and Hausdorff. Hence $\overline{U}$ is T_3. Also, U is open in $\overline{U}$. So that by the Theorem 40.6, there is an open set W in $\overline{U}$ such that $x \in W \subset \mathrm{cl}_{\overline{U}}(W) \subset U$. But then U, W are open in $\overline{V}$ implies $W = U \cap W$ is open in U. Since U is open in X, W is open in X. Also $\mathrm{cl}_{\overline{U}}(W)$ is closed in $\overline{U}$, and hence, compact. Thus $x \in W \subset \overline{W} \subset \mathrm{cl}_{\overline{U}}(W) \subset U \subset V$. Thus $\overline{W} \in \mathscr{B}_x$. Hence $\mathscr{B}_x$ is a neighbourhood base at x. Converse is at once. □

The following is a consequence of the above theorem and the Corollary 40.7.

49.4. Corollary. *a locally compact Hausdorff space is regular.*

49.5. Theorem. *Let X be a locally compact Hausdorff space. Then*

(1) Intersection of two locally compact subsets of X is locally compact.
(2) A closed subset of X is locally compact.
(3) An open subset of X is locally compact.

PROOF. ☞(1) Let $A, B \subset X$ be locally compact and $x \in A \cap B$. Then there are $C \subset A, D \subset B$ such that C is open in A, D is open in B and $\mathrm{cl}_A(C), \mathrm{cl}_B(D)$ are compact. Thus $x \in C \cap D$, and $C \cap D$ is an open set in $A \cap B$. Also, $C \cap D \subset C$, and hence, $\mathrm{cl}_{A\cap B}(C \cap D) \subset \mathrm{cl}_A(C)$ is compact.

☞(2) Let $Y \subset X$ be closed and $x \in Y$. Since X is locally compact, there is a neighbourhood U of x in X such that $\overline{U}$ is compact. But then $x \in U \cap Y \subset \overline{U} \cap Y$, $U \cap Y$ is open and $\overline{U} \cap Y$ is closed in Y. Thus $x \in U \cap Y \subset \mathrm{cl}_Y(U \cap Y) \subset \overline{U} \cap Y$. Also, $\overline{U} \cap Y$ is closed in $\overline{U}$, and hence, is compact. Thus Y is locally compact.

☞(3) Let $Y \subset X$ be open and $x \in Y$. Since X is locally compact, it is T_3. So, there is an open set H in X such that $x \in H \subset \overline{H} \subset Y$. On the other hand, by the local compactness of X, there is an open set U such that $x \in U$ and $\mathrm{cl}_X(U)$ is compact. Clearly, $U \cap H$ is an open neighbourhood of x in Y. Note that $\mathrm{cl}_X(U \cap H) \subset \mathrm{cl}_X(V) \subset Y$. Using this we get, $\mathrm{cl}_Y(U \cap H) = \mathrm{cl}_X(U \cap H) \cap Y = \mathrm{cl}_X(U \cap H) \subset \mathrm{cl}_X(U)$, which is compact. Thus $\mathrm{cl}_X(U \cap H)$ is compact, That is $\mathrm{cl}_Y(U \cap H)$ is compact. Thus $\mathrm{cl}_Y(U \cap H)$ is a compact neighbourhood of x in Y. This completes the proof. □

49.6. Corollary. *Let X be a locally compact Hausdorff space, $Y \subset X$ be open and $Z \subset X$ be closed subsets of X. Then $Y \cap Z$ is locally compact.*

The interesting part is the converse of the above corollary. However, before that we prove a technical lemma.

49.7. Lemma. *Let X be a Hausdorff space and $Y \subset X$ be a dense and locally compact. Then Y is open in X.*

PROOF. Let $y \in Y$ and U be an open set in Y such that $y \in U$ and $\mathrm{cl}_Y(U)$ is compact. That is, $\mathrm{cl}_X(U) \cap Y$ is compact. Since X is Hausdorff, $\mathrm{cl}_X(U) \cap Y$ is closed in X. Note that $U \subset Y$, and hence, $U \subset \mathrm{cl}_X(U) \cap Y$, a closed set in X. So, $\mathrm{cl}_X U \subset \mathrm{cl}_X(U) \cap Y \subset Y$. Now U is open in Y. So, there is an open set V in X such that $U = V \cap Y$. Since $V \cap Y \subset V$, $\mathrm{cl}_X(V \cap Y) \subset \mathrm{cl}_X(V)$. On the other hand, let $x \in \mathrm{cl}_X(V)$ and G be an open set containing x. Then $G \cap V \neq \emptyset$. Since Y is dense in X, the open set $G \cap V$ contains a point of Y. But then G contains a point of $Y \cap V$. That is, $x \in \mathrm{cl}_X(Y \cap V)$. Thus $\mathrm{cl}_X(V) = \mathrm{cl}_X(V \cap Y)$. Hence $V \subset \mathrm{cl}_X(V) = \mathrm{cl}_X(Y \cap V) = \mathrm{cl}_X(U) \subset \mathrm{cl}_X(U) \cap Y \subset Y$. Thus $V \subset Y$. That is $y \in V \subset Y$. Hence Y is open. □

49.8. Theorem. *Let X be a Hausdorff space. Let $E \subset X$ be locally compact. Then there are $Y \subset X$, an open set, and $Z \subset X$, a closed set in X such that $E = Y \cap Z$.*

PROOF. Clearly, $\overline{E}$ is Hausdorff and E is a dense subset of $\overline{E}$. Hence E is open in $\overline{E}$. So, there is an open set $Y \subset X$ such that $E = Y \cap \overline{E}$. Taking $Z = \overline{E}$, the result follows. □

49.9. Theorem. *A continuous open image of a locally compact space is locally compact.*

PROOF. Let X be a locally compact space, Y be a topological space and $f : X \to Y$ be an continuous open map. Let $y \in Y$ and $x \in X$ such that $f(x) = y$. Let $U \in \mathscr{U}_x$ be compact. Then $y \in f(U^\circ) \subset f(U)$. Also, $f(U)$ is compact and $f(U^\circ)$ is open. Thus Y is locally compact. □

49.10. Theorem. *Let $\{X_i : i \in I\}$ be a family of topological spaces. Then $X = \prod_{i\in I} X_i$ is locally compact if and only if (1) each X_i is locally compact and (2) all but finitely many of X_i are compact.*

PROOF. Assume that X is locally compact. Since each projection is continuous and open, it follows that each X_i is locally compact. Let $\mathbf{x} \in X$ and $U \in \mathscr{U}_{\mathbf{x}}$ be compact. Then all but finitely many $X_i = \pi_i(U)$. Hence all but finitely many X_i are compact.

Conversely, suppose each X_i is locally compact and all but finitely many of X_i are compact. Let $i_1, i_2, \ldots, i_n$ be such that $X_{i_1}, X_{i_2}, \ldots, X_{i_n}$ be noncompact locally compact space. Let $\mathbf{x} \in X$ and $U_{i_1}, U_{i_2}, \ldots, U_{i_n}$ be compact neighbourhoods of $\mathbf{x}_{i_1}, \mathbf{x}_{i_2}, \ldots, \mathbf{x}_{i_n}$ respectively. Then $\cap_{k=1}^n \pi_{i_k}^{-1}(U_{i_k})$ is a compact neighbourhood of $\mathbf{x}$. This completes the proof. □

50. Compactification

50.1. Definition. Let $(X, \mathscr{T})$ be a topological space. A compact topological space $\widetilde{X}$ is said to be a *compactification of* X if X can be embedded as a dense subset of $\widetilde{X}$.

50.2. Remark. Thus compactification of a topological space X is a pair (Y, h), where Y is a compact topological space, $h\colon X \to Y$ is an into homeomorphism such that $h(X)$ is dense in Y.

We begin with a well known compactification known as *the one point compactification*.

50.3. Theorem. *Let X be a locally compact, noncompact Hausdorff topological space. Let $Y = X \cup \{\infty\}$, where ∞ is just a symbol. Define $G \subset Y$ to be open if either $G \subset X$ and G is open or $\infty \in G$ and $Y \setminus G$ is compact in X. Then*

(1) Y is compact.
(2) Y is Hausdorff.
(3) X is open in Y.
(4) The topology of X coincides with the subspace topology on X.
(5) X is dense in Y.

Proof. First we show that the open sets, so defined, constitute a topology on Y. Clearly, $\emptyset$ is open in X, so it is open in Y. Also, $Y \setminus Y$ is compact in X. Thus Y is open too.

Let $\{G_i : i \in I\}$ be a family of open sets in Y. Let $G = \cup\{G_i : G_i \text{ is open in X}\}$. Clearly, G is open in X. Let $H = \cup\{G_i : \infty \in G_i\}$. Clearly, for all such G_i, $Y \setminus G_i$ is compact in X. Since X is Hausdorff, by the Lemma 47.11, $\cap\{Y \setminus G_i : \infty \in G_i\}$ is compact in X. That is, $\cup\{G_i : \infty \in G_i\}$ is open in Y. Now $Y \setminus (G \cup H) \subset Y \setminus H$, a compact set. Also $Y \setminus (G \cup H) = (Y \setminus G) \cap (Y \setminus H) = (X \setminus G) \cap (Y \setminus H)$, because $\infty \in H$. But G is open in X, hence $X \setminus G$ is closed in X and $Y \setminus H$ is compact. Thus $(X \setminus G) \cap (Y \setminus H)$ is a closed subset of $Y \setminus H$. Thus $Y \setminus (G \cup H) = (X \setminus G) \cap (Y \setminus H)$ is compact. Thus $G \cup H$ is open in Y. This proves that the open sets defined in the hypothesis, form a topology on Y. Now let $\mathscr{A}$ be an open cover for Y. Fix $G \in \mathscr{A}$ such that $\infty \in G$. Then $\mathscr{A} \setminus \{G\}$ is an open cover for $Y \setminus G$, and hence, has a finite subcover $\mathscr{A}'$ for $Y \setminus G$. That is, $\mathscr{A}' \cup \{G\}$ is a finite open cover for Y.

Since X is T_2, two points of X can be separated by means of two open sets in X, and hence, open in Y. Now let $x \in X$. Since X is locally compact, there is a compact neighbourhood U of x in X. But then U° and $Y \setminus U$ are two disjoint open sets in Y containing x and ∞ respectively. Thus Y is T_2. The same argument shows that ∞ is closed, and hence, X is open in X.

Since every open subset of X is also open in Y, it is clear that the subspace topology is finer than the topology on X. On the other hand, let $U \subset X$ be open in the subspace topology. So, there is $G \subset Y$ such that $U = G \cap X$, where either G is open in the original topology of X or $Y \setminus G$ is compact. In the first

case, we are done, In the second case, $Y \setminus G$ will be a closed subset of X and so, $U = X \setminus (Y \setminus G)$ is open in X.

Now suppose $\{\infty\}$ is open in Y. So, $X \setminus \{\infty\}$ is compact in X, which is not the case. Thus every open set containing ∞ also contains a point of X. Thus X is dense in Y. □

50.4. Corollary. *A locally compact T_2-space is completely regular.*

Proof. Indeed a locally compact T_2-space is embedded into a compact Hausdorff space, and hence, it is a subspace of a completely regular space. Hence it is completely regular. □

We add two more facts in the following theorems.

50.5. Theorem. *Let X be a locally compact, noncompact Hausdorff topological space. Let ∞, ∞' be two symbols and $Y = X \cup \{\infty\}$ and $Z = X \cup \{\infty'\}$ be the topological spaces obtained as in the Theorem 50.3. Then $f : Y \to Z$ defined by $f(x) = x$ if $x \in X$ and $f(\infty) = \infty'$ is a homeomorphism.*

Proof. Let $G \subset X$ be open in Y. So, G is open in X, and hence, G is open in Z. Now let $\infty \in G$ and G be open in Y. So, $Y \setminus G$ is compact in X, and hence, $f(G) = G \cup \{\infty'\} \setminus \{\infty\}$ is open in Z. Thus f is an open map. The symmetry of the argument shows that f is continuous. □

50.6. Definition. The compactification of a locally compact Hausdorff space X obtained in the Theorem 50.3 is called the *one point compactification* of X. It is also known as the *Alexandroff compactification* of X

The Theorem 50.3 states that the one point compactification is unique up to homeomorphism.

The following is the converse to the Theorem 50.3.

50.7. Theorem. *Let X be a topological space, Y be a compact Hausdorff topological space and $h : X \to Y$ be an embedding such that $h(X)$ is dense in Y and $Y \setminus h(X)$ is singleton. Then X is locally compact and Hausdorff.*

Proof. Indeed being homeomorphic to a subspace of a Hausdorff space, X is Hausdorff. Let $Y \setminus h(X) = \{a\}$, and $x \in X$. Since Y is T_2, there are two disjoint open sets U, V in Y such that $x \in U, a \in V$. But then $x \in U \subset Y \setminus V$ and $Y \setminus V$, being a closed subspace of a compact space, is compact. Thus $x \in U \subset \overline{U} \subset Y \setminus V$, so that $\overline{U}$ is a compact neighbourhood of x in X. This completes the proof. □

50.8. Example.

(1) The one point compactification of $\mathbb{R}$ as well as any open interval in $\mathbb{R}$ is homeomorphic to the unit circle
$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$
(2) The one point compactification of $\mathbb{N}$ is $\{\frac{1}{n}\} \cup \{0\}$.
(3) The one point compactification of $[0, 1)$ is $[0, 1]$.

50.9. Theorem. *Let X be a locally compact T_2-space. For each $f \in C_b(X, \mathbb{R})$, let $a_f = \inf\{f(x) : x \in X\}, b_f = \sup\{f(x) : x \in X\}$ and $J_f = [a_f, b_f]$. Let $e\colon X \to \prod_{f \in C_b(X,\mathbb{R})} J_f$ be the evaluation map. Then $\overline{e(X)}$ is a compactification of X.*

Proof. The proof is combination of various results already proved. □

50.10. Definition. Let X be a locally compact Hausdorff space. Then the compactification $\overline{e(X)}$ of X mentioned in the Theorem 50.9 is called the *Stone-Čech compactification* of X and is denoted by $\beta(X)$.

CHAPTER 9

Connectedness

51. Connected Spaces

51.1. Definition. Let X be a topological space. X is said to be *disconnected* if there are two subsets U, V of X satisfying the following.

(1) $U \neq \emptyset \neq V$.
(2) U, V are open in X.
(3) $U \cup V = X$.
(4) $U \cap V = \emptyset$.

Such a pair U, V of subsets of X is said to be a *separation* of X. If X is not disconnected, then it is called *connected*.

51.2. Remark. Both singleton and the empty set cannot have two nonempty proper subsets, and hence, they are connected.

51.3. Theorem. *The following are equivalent for a topological space X.*

(1) X is disconnected.
(2) X has a nonempty proper clopen subset.
(3) Characteristic function of some nonempty proper subset of X is continuous.
(4) There is a nonconstant continuous function from X to $\{0, 1\}$.

Proof. ☞(1) ⇒ (2) Assume (1). Since X is disconnected, it has a separation U, V. $U \cap V = \emptyset$ and $U \cup V = X \Rightarrow U = X \setminus V$. Also, $U \neq \emptyset$. Further $V \neq \emptyset \Rightarrow U \neq X$. Also U is open. Finally, V is open $\Rightarrow U$ is closed. Thus $U \subset X$ is a nonempty clopen subset.

☞(2) ⇒ (3) Assume (2). Let $U \subset X$ be a nonempty proper clopen subset. Consider $\chi_U \colon X \to \{0, 1\}$. Then $\chi_U^{-1}\{0\} = V = X \setminus U$, an open set because U is closed and $\chi_U^{-1}\{1\} = U$, an open set. Thus χ_U is continuous.

☞(3) ⇒ (4) The function in (3) is nonconstant and continuous.

☞(4) ⇒ (1) Assume (4). Let $f : \{0,1\}$ be a nonconstant continuous function $f\colon X \to \{0,1\}$. Define $U = f^{-1}(1), V = f^{-1}(0)$. Since f is continuous, U, V are open. Since f is onto, U, V are nonempty. Also $U \cup V = f^{-1}(1) \cup f^{-1}(0) = f^{-1}(\{0,1\}) = X$. Also, $U \cap V = f^{-1}(1) \cap f^{-1}(0) = f^{-1}(\{0\} \cap \{1\}) = f^{-1}(\emptyset) = \emptyset$. Thus X is disconnected. □

51.4. Example. Every locally constant function on a connected space is constant.

Indeed, let X be a connected space Y be a topological space and $f\colon X \to Y$ be a locally constant function. Suppose f is not constant. Without loss of generality, we assume that f is onto. For every $y \in Y$, let $U_y = f^{-1}y$. Clearly, for every $x \in U_y$, there is an $G \in \mathscr{U}_x$ such that $f(G)$ is singleton. Thus $x \in U_y^\circ$. Thus U_y is open. Consequently, $\{U_y : y \in Y\}$ is a partition of X by means of open sets. But then U_y is clopen. This shows that $X = U_y$, showing that f is continuous.

51.5. Theorem. *Let X be an ordered topological space. Let $a \in X$ have the immediate successor $b \in X$. Then X is disconnected. In particular, a well-ordered non-singleton is disconnected.*

Proof. Clearly, $(a, \infty) = [b, \infty)$ is a nonempty proper clopen subset of X. Hence X is disconnected. □

51.6. Theorem. *A continuous image of a connected space is connected.*

Proof. Let X be a connected topological space, Y be a topological space and $f\colon X \to Y$ be a continuous function. Suppose Y is disconnected. So, there is an onto continuous function $g\colon Y \to \{0,1\}$. Hence $g \circ f\colon X \to \{0,1\}$ is an onto continuous function. Thus X is disconnected, a contradiction. Thus Y must be connected. □

51.7. Corollary. *Let $\mathscr{T}_1, \mathscr{T}_2$ be two topologies on a set X such that $\mathscr{T}_1 \leq \mathscr{T}_2$. If $(X, \mathscr{T}_2)$ is connected, then so is $(X, \mathscr{T}_1)$.*

Proof. Indeed, $(X, \mathscr{T}_1)$ is a continuous image of $(X, \mathscr{T}_2)$ under the identity map $\iota_X\colon (X, \mathscr{T}_2) \to (X, \mathscr{T}_1)$. Hence the result follows. □

51.8. Corollary. *Homeomorphic image of a connected space is connected.*

51.9. Corollary. *Let X be a topological space and $Z \subset Y \subset X$. Z is connected as a subspace of Y if and only if Z is connected as a subspace of X.*

Proof. Clearly, $\iota\colon Z \to X$ is continuous, and hence, the result follows. □

51.10. Theorem. *Let X be a topological space and $A \subset X$ be connected. Let $A \subset B \subset \overline{A}$. Then B is connected.*

PROOF. Suppose, if possible, that B is disconnected. Let $f\colon B \to \{0,1\}$ be an onto continuous function. Let $x, y \in B$ such that $f(x) = 0, f(y) = 1$. Then $f^{-1}(\{0\})$ is an open set in B containing x. So, there is $a \in A$ such that $a \in f^{-1}(\{0\})$. In the same way, we get $b \in A$ such that $f^{-1}(\{1\})$. Thus $f_{|_A}$ is a nonconstant continuous function from A to $\{0,1\}$ making A disconnected. This cannot happen. Thus B must be connected. □

51.11. Corollary. *Closure of a connected set is connected.*

51.12. Example. Boundary of a connected space need not be connected. Indeed, $(0,1)$ is connected but its boundary is $\{0,1\}$, a disconnected set.

51.13. Example. Derived set of a connected set may be disconnected. Indeed, consider the VIP topology $\mathscr{T}_0$ on $\mathbb{R}$. Let $0 \in A \subset \mathbb{R}$. Then A is connected. Indeed suppose $B \subsetneq A$ is clopen in A. Since B is open, $0 \in B$ and since B is a proper closed subset, $0 \notin B$, a contradiction. Thus A is connected. In particular, $A = \{0\}$ is connected but $A' = \mathbb{R} \setminus \{0\}$ is discrete, and hence, disconnected.

However, in a T_1-space, the scenario is different.

51.14. Theorem. *Let X be a T_1-space and $A \subset X$ be connected and non-singleton. Then $A \subset A'$.*

PROOF. Let $x \in A$. Suppose $x \notin A'$. So, there is an open set $G \subset X$ such that $G \cap A = \{x\}$. Thus $\{x\}$ is open in A. On the other hand, A is T_1, and hence, $\{x\}$ is closed in A. Thus $\{x\}$ is a nonempty, proper clopen subset of A, a contradiction. Thus $x \in A'$. □

51.15. Corollary. *Let X be a T_1-space and $A \subset X$ be connected and non-singleton. Then A' is connected.*

PROOF. Since A is a connected non-singleton subset of a T_1-space, $A \subset A' \subset \overline{A}$. Hence A' is connected. □

51.16. Corollary. *Let X be a T_1-space and $A \subset X$ be connected and non-singleton. Then $\overline{A}$ is perfect.*

PROOF. Clearly, $\overline{A}$ is connected and non-singleton. Since $\overline{A}$ is closed $\overline{A}' \subset \overline{A}$. Also, by Theorem 51.14, $\overline{A} \subset \overline{A}'$. Thus $\overline{A}' = \overline{A}$. That is $\overline{A}$ is perfect. □

51.17. Theorem. *For $a, b \in \mathbb{R}$ with $a < b$, the open interval (a, b) is connected.*

PROOF. Let $f : (a, b) \to \{0, 1\}$ be a continuous function. Suppose, if possible, that there are $a_1, b_1 \in (a, b)$ such that $f(a_1) = 0, f(b_1) = 1$. Without loss of generality, we assume that $a_1 < b_1$. Let $A = \{x : x > a_1, f(x) = 1\}$. Clearly, A is bounded below by a_1. Let $c = \inf A$. Since f is continuous, $c \in \text{Int}(f^{-1}(\{f(c)\}))$. Let $c \in (d, e) \subset f^{-1}(\{f(c)\})$.

Suppose $f(c) = 0$. Since $\frac{c+e}{2}$ is not a lower bound, there is $\xi \in (c, \frac{c+e}{2})$ such that $\xi \in A$. But then $1 = f(\xi) \in f((d, e)) = \{0\}$, a contradiction.

Suppose $f(c) = 1$. But then $d < \frac{c+d}{2} < c$. Thus $0 = f(\frac{c+d}{2}) \in f((d, e)) = \{1\}$, a contradiction.

Thus $f(c)$ is not well-defined. Thus f must be constant. This proves that (a, b) is connected. □

51.18. Corollary. *For $-\infty \leq a < b \leq \infty$, $(a, b), [a, b], [a, b), (a, b]$, whenever they are defined, are connected.*

PROOF. The proof follows from the Theorem 51.10 and the Theorem 51.17. □

51.19. Theorem. *Suppose $A \subset \mathbb{R}$ is neither an interval nor a singleton. Then A is disconnected.*

PROOF. Since A is not a singleton, A contains at least two points. Also, since A is not an interval, there are $a, b, c \in \mathbb{R}$ such that for some $a < c < b$, $a, b \in A$ but $c \notin A$. But then $U = (-\infty, c) \cap A, V = (c, \infty) \cap A$ form a separation of A, and hence, A is disconnected. □

51.20. Corollary. *$I \subset \mathbb{R}$ is connected if and only if I is $\emptyset$ or a singleton or an interval.*

51.21. Theorem. *Let U, V be a separation of a topological space X and $A \subset X$ be connected. Then either $A \subset U$ or $A \subset V$.*

PROOF. Note that U is a clopen subset of X, and hence, $U \cap A$ is a clopen subset of A. Since A is connected, $U \cap A = \emptyset$ or $U \cap A = A$. That is, $A \subset V$ of $A \subset U$. □

51.22. Theorem. *Let X be a topological space and $\{A_i : i \in I\}$ be a family of connected subsets of X. If $\cap A_i \neq \emptyset$, then $\cup A_i$ is connected.*

PROOF. let $a \in \cap A_i$. Let $A = \cup A_i$. Let $f\colon A \to \{0,1\}$ be a continuous function and $f(a) = 0$. Let $b \in A$. Then there is $j \in I$ such that $b \in A_j$. But then $a, b \in A_j$. Since $f_{|_{A_j}}$ is constant $f(b) = f(a) = 0$. Thus f is a constant function. This proves that A is connected. □

51.23. Theorem. *Let X be a topological space and $\{A_n : n \in \mathbb{N}\}$ be a family of connected subsets of X. If $A_n \cap A_{n+1} \neq \emptyset$ for every $n \in \mathbb{N}$, then $\cup A_n$ is connected.*

PROOF. Applying the Theorem 51.22 repeatedly and proving inductively, we get $B_n = \cup_{k=1}^{n} A_k$ is connected for every $n \in \mathbb{N}$. Since $\cap_{n=1}^{\infty} B_n = A_1 \neq \emptyset$, $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$ is connected. □

51.24. Theorem. *Let X be a topological space. Define $\sim$ on X by $x \sim y$ if there is a connected subset A of X such that $x, y \in A$. Then $\sim$ is an equivalence relation on X.*

PROOF. Since each singleton is connected, $\sim$ is reflexive. If a connected set contains x, y then it also contains y, x, and hence, $\sim$ is symmetric too. Finally, suppose $x \sim y$ and $y \sim z$. Let $A, B \subset X$ be connected and $x, y \in A, y, z \in B$. But then $A \cap B \neq \emptyset$. Hence $A \cup B$ is connected. Since $x, z \in A \cup B$, $x \sim z$. □

51.25. Definition. Let X be a topological space. An equivalence class of the relation defined in the Theorem 51.24 is called *component* of X.

51.26. Theorem. *Let X be a topological space. A component of X is a maximal connected subset of X.*

PROOF. Let C be a component of X and $x \in C$. For every $y \in C$, let $A_{x,y}$ be a connected set containing x and y. Since $x \in \cap_{y \in C} A_{x,y}$, $C = \cup_{y \in C} A_{x,y}$ is connected. Suppose D is a connected set and $C \subset D$. Let $z \in D$. Then D is a connected set containing x, z and hence, $x \sim z$. Thus $z \in C$. This proves that C is a maximal connected set. □

51.27. Theorem. *A component of a topological space is closed.*

PROOF. Let X be a topological space and C be its component. Since C is connected, so is $\overline{C}$. But the maximality of C ensures that $C = \overline{C}$. Thus C is closed. □

51.28. Lemma. *Let $\{X_i : i \in I\}$ be a family of connected spaces. Let $\mathbf{a} \in X = \prod_{i\in I} X_i$. Let*

$$Y_{\mathbf{a}} = \{\mathbf{x} \in X : \mathbf{a}_i = \mathbf{x}_i \text{ for all but finitely many } i \in I\}.$$

Then $Y_{\mathbf{a}}$ is connected.

PROOF. Let $\mathbf{x} \in Y_{\mathbf{a}}$ and $F = \{j_1, j_2, j_3, \ldots, j_n\} \subset I$ be such that $\mathbf{x}_j \neq \mathbf{a}_j$ for every $j \in F$. Define

$$\begin{aligned}
A_1 &= \prod_{\substack{j\in I\\ j\notin F}} \{\mathbf{a}_j\} \times \ X_{j_1} \ \times \{\mathbf{a}_{j_2}\} \times \{\mathbf{a}_{j_3}\} \times \cdots \times \{\mathbf{a}_{j_{n-1}}\} \times \{\mathbf{a}_{j_n}\}\\
A_2 &= \prod_{\substack{j\in I\\ j\notin F}} \{\mathbf{a}_j\} \times \{\mathbf{x}_{j_1}\} \times \ X_{j_2} \ \times \{\mathbf{a}_{j_3}\} \times \cdots \times \{\mathbf{a}_{j_{n-1}}\} \times \{\mathbf{a}_{j_n}\}\\
A_3 &= \prod_{\substack{j\in I\\ j\notin F}} \{\mathbf{a}_j\} \times \{\mathbf{x}_{j_1}\} \times \{\mathbf{x}_{j_2}\} \times \ X_{j_3} \ \times \cdots \times \{\mathbf{a}_{j_{n-1}}\} \times \{\mathbf{a}_{j_n}\}\\
\cdots &= \quad \cdots \quad \times \ \cdots \ \times \ \cdots \ \times \ \cdots \ \times \cdots \times \quad \cdots \quad \times \ \cdots\\
A_n &= \prod_{\substack{j\in I\\ j\notin F}} \{\mathbf{a}_j\} \times \{\mathbf{x}_{j_1}\} \times \{\mathbf{x}_{j_2}\} \times \{\mathbf{x}_{j_3}\} \times \cdots \times \{\mathbf{x}_{j_{n-1}}\} \times \ X_{j_n}
\end{aligned}$$

and $A_{\mathbf{x}} = A_1 \cup A_2 \cup \cdots \cup A_n$. We note the following.

(1) For every $1 \leq k \leq n$, A_k is of the form $X(\mathbf{y}, i)$ for some $\mathbf{y} \in X$ and for some $i \in I$.
(2) Since $X(\mathbf{y}, i)$ is homeomorphic to X_i, $X(\mathbf{y}, i)$ is connected. Thus A_k is connected for every $1 \leq k \leq n$.
(3) $A_k \cap A_{k+1} \neq \emptyset$ for every $1 \leq k \leq n-1$.
(4) Applying the Theorem 51.23, we conclude that $A_{\mathbf{x}}$ is connected.
(5) $\mathbf{a}, \mathbf{x} \in A_{\mathbf{x}}$.

Since $Y_{\mathbf{a}} = \cup_{\mathbf{x}\in Y_{\mathbf{a}}} A_{\mathbf{x}}$, $Y_{\mathbf{a}}$ is connected. □

51.29. Theorem. *Let $\{X_i : i \in I\}$ be a family of topological spaces. Let $\mathbf{a} \in X = \prod_{i\in I} X_i$. Then X is connected if and only if each X_i is connected.*

PROOF. If X is connected, then X_i, being a continuous image of X, is connected. So, assume that each X_i is connected. Let $\mathbf{a} \in X$. Then as in the Lemma 51.28, $Y_{\mathbf{a}}$ is connected. We show that $Y_{\mathbf{a}}$ is dense in X. Let $U = \cap_{k=1}^{n} \pi_{i_k}^{-1}(U_{i_k})$, be a basic open set with U_{i_k} open in X_{i_k} for every $1 \leq k \leq n$. Choose $\xi_{i_k} \in U_{i_k}$. Define

$$\mathbf{x}_i = \begin{cases} \xi_{i_k}, & \text{if } i = i_k \text{ for some } k;\\ \mathbf{a}_i, & \text{if } i \neq i_k \text{ for any } k. \end{cases}$$

Then $\mathbf{x} \in Y_{\mathbf{a}} \cap U$. Thus $Y_{\mathbf{a}} \cap U \neq \emptyset$. Thus X is connected. □

51.30. Theorem. *A connected normal space is either a singleton or uncountable.*

PROOF. Let X be a connected normal space. Let x, y be two distinct elements of X. Then $\{x\}$ and $\{y\}$ are two disjoint closed subsets of X. So, there is a continuous function $f\colon X \to [0,1]$ such that $f(x) = 1$ and $f(y) = 0$. Since X

is connected, so is $f(X)$. So, $f(X) \subset [0,1]$ is an interval containing $0, 1$. Hence $f(X) = [0,1]$. Since $[0,1]$ is uncountable, so must be X. □

51.31. Remark. The above theorem is not true for a T_4-space. Indeed $\{0,1\}$ with the indiscrete topology is T_4 and connected.

51.32. Corollary. *A connected metric space is either singleton or uncountable.*

51.33. Definition. A topological space X is said to be *totally disconnected* if each of its components is singleton.

51.34. Example. Cofinite topology on an infinite set is connected. Indeed, a proper subset of a cofinite topological space is closed if and only if it is finite, which is not open as the whole set is infinite.

51.35. Example. The argument similar to the above shows that an uncountable cocountable space is connected.

51.36. Example. $\mathbb{R}_\ell$ is totally disconnected. Let $Y \subset \mathbb{R}$ have at least two points, say $a, b \in Y$ with $a < b$. Then $A = [b, \infty) \cap Y$ is a proper clopen subset of Y. Thus no subset of $\mathbb{R}_\ell$ having at least two points is connected.

51.37. Example. A discrete space with at least two points is totally disconnected. Indeed, let X be a discrete space. Let $A \subset X$ and $x, y \in A$ be distinct. Then $\chi_{\{x\}|_A}$ is a nonconstant continuous function from A to $\{0,1\}$. Thus no subset of X other than singletons is connected. Thus X is totally disconnected.

51.38. Example. $\mathbb{Q}$ is totally disconnected and non-discrete. Indeed $\mathbb{Q}$ is not discrete. On the other hand, Let $A \subset \mathbb{Q}$ be connected. Then $A \subset \mathbb{R}$ is connected. Thus A must be a singleton or an interval. Since $A \subset \mathbb{Q}$, A must be singleton. Thus components of $\mathbb{Q}$ are singletons.

51.39. Example. The Cantor ternary set C is totally disconnected. From the construction of Cantor set in the Example 27.14, it is clear that I_n consists of union of closed intervals of length 3^{-n}. Let $x, y \in C$ and $r = |x - y|$. Choose $n \in \mathbb{N}$ such that $3^{-n} < r$. Then x and y cannot be in the same interval in the expression of I_n. Thus there is $z \notin I_n$ such that $x < z < y$. Thus $(-\infty, z) \cap C$ and (z, ∞) is a separation of C that also separates x, y. Thus C is totally disconnected.

52. Path Connected Spaces

52.1. Definition. Let X be a topological space and $x, y \in X$. A *path from x to y* is a continuous function $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. In this case, we write $x \overset{p}{\sim} y$.

52.2. Theorem. *Let X be a topological space. $\overset{p}{\sim}$ is an equivalence relation.*

PROOF. For $x \in X$, define $\gamma\colon [0,1] \to X$ by $\gamma(t) = x$, $(t \in [0,1])$. Then being a constant function, γ is continuous with the required conditions.

$\gamma\colon [0,1] \to X$ be a path from x to y, then $\theta(t) = \gamma(1-t)$, $(t \in [0,1])$, is a path from y to x. Thus $\overset{p}{\sim}$ is symmetric.

Finally, let $\gamma, \theta\colon [0,1] \to X$ be two continuous functions such that $\gamma(0) = x, \gamma(1) = y = \theta(0), \theta(1) = z$. Define $\eta\colon [0,1] \to X$ by

$$\eta(t) = \begin{cases} \gamma(2t), & \text{if } 0 \le t \le \frac{1}{2}; \\ \theta(2t-1), & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Since $t \in [0, \frac{1}{2}] \mapsto \gamma(2t)$ and $t \in [\frac{1}{2}, 1] \mapsto \theta(2t-1)$ are continuous and since they agree on the point of intersection of their domains, η is continuous by the Pasting Lemma. Also, $\eta(0) = \gamma(0) = x, \eta(1) = \theta(1) = z$. $\overset{p}{\sim}$ is transitive. This completes the proof. □

52.3. Definition. Let X be a topological space. For $x \in X$, $\{y \in X : x \overset{p}{\sim} y\}$ is called the *path component* of X containing x. X is said to be *path connected* if there is a path between every pair of points of X, that is, if $x \overset{p}{\sim} y$ for all $x, y \in X$, that is if X has only one path component.

52.4. Corollary. *Let $A, B \subset X$ be path connected and $A \cap B \neq \emptyset$. Then $A \cup B$ is path connected.*

PROOF. Let $a \in A, b \in B$. Then $a \overset{p}{\sim} x$ and $x \overset{p}{\sim} b$. So, $a \overset{p}{\sim} b$. Thus $A \cup B$ is path connected. □

52.5. Example. Let X be either $\mathbb{R}^n$ or $\mathbb{C}^n$ with the usual topology. For $x, y \in X$, $\gamma(t) = ty + (1-t)x$, $(t \in [0,1])$ defines a path from x to y.

52.6. Lemma. *Let X be a topological space and $x, y \in X$. If $x \overset{p}{\sim} y$, then $x \sim y$.*

PROOF. For a path γ from x to y, $\gamma([0,1])$ is a connected subset of X containing x, y. So, $x \sim y$. □

52.7. Theorem. *A path connected space is connected.*

PROOF. Let X be a path connected space. Let $x, y \in X$. Then $x \overset{p}{\sim} y \Rightarrow x \sim y$. Thus X is connected. □

The converse to the above theorem is not true.

52.8. Example. Let $S_1 = \{(x, \sin\frac{1}{x}) : 0 < x \le 1\}$, $S_2 = \{(0, y) : -1 \le y \le 1\}$ and $S = S_1 \cup S_2$. Then S is connected but not path connected. Indeed let $a = (x_1, \sin\frac{1}{x_1})$ and $b = (x_2, \sin\frac{1}{x_2})$ be in S_1. define $f\colon [0,1] \to S_1$ by $f(t) =$

$((1-t)x_1+tx_2, \sin(\frac{1}{(1-t)x_1+tx_2}))$. Then f is a path joining a and b in S_1. Thus S_1 is path connected, and hence, connected. Now let $(0,y) \in S_2$. Let $t \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. Let $x_n = \frac{1}{t+2\pi n}$, $n \in \mathbb{N}$. Then $a_n = (x_n, \sin(\frac{1}{x_n})) \in S_1$ and $a_n \to (0,y)$. Thus $S \subset \overline{S_1}$. This ensures that S is path connected. On the other hand, suppose $g\colon [0,1] \to S$ is a path such that $g(0) = (0, \frac{1}{2})$ and $g(1) = (1, \sin(1))$. Applying the projection maps, as in the Theorem 22.9, we can write $g(t) - (g_1(t), g_2(t))$. Also, in this case, g_1, g_2 are continuous. Clearly, $g_1(0) = 0$ and $g_1(1) = 1$. Applying the intermediate theorem, there is $0 < s_1 < 1$ such that $g_1(s_1) = \frac{2}{\pi}$, repeating the application of the intermediate theorem, we get $0 < s_2 < s_1$ such that $g_1(s_2) = \frac{2}{3\pi}$. Extending the same argument inductively, we get a decreasing sequence $1 > s_1 > s_2 > s_3 > \cdots$ such that $g_1(s_n) = \frac{2}{(2n-1)\pi}$. Being a decreasing sequence bounded below by 0, $s_n \to a$ for some $a \in [0,1]$. So, $g(t_n) \to g(a)$. So that $g_2(t_n) \to g_2(a)$. But $g_2(t_n) = (=1)^{n+1}$, which does not converge. Thus such a g does not exists. This space is referred to as *Topologist's Sine Curve*.

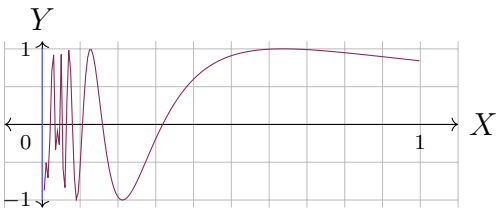


FIGURE 35. Topologist Sine Curve

52.9. Remark. The above Example shows that closure of a path connected space need not be path connected.

52.10. Theorem. *A finite connected space is path connected.*

PROOF. Let $X = \{x_1, x_2, \ldots, x_n\}$ Let $C_1, C_2, \ldots, C_j$ be the path components of X. If each C_i is closed, $C_2 \cup C_3 \cup \cdots \cup C_n$ is closed too. But then C_1 is clopen, a contradiction. Thus some C_i is not closed. Without loss of generality, we assume that C_1 is not closed. Let x_i be a limit point of C_1 such that $x_i \notin C_1$. Let U be the intersection of all open sets containing x_i. Then U is a finite intersection of open sets. Hence U is the smallest open set containing x_i. Let $x_k \in U \cap C_1$. Define $f\colon [0,1] \to X$ by $f(0) = x_i$ and $f(t) = x_k$ if $0 < t \leq 1$. Let V be an open subset of X. If $x_i \in V$, then $x_k \in V$, and hence, $f^{-1}(V) = [0,1]$. If $x_k \in V$ but $x_i \notin V$, then $f^{-1}(V) = f^{-1}(\{x_k\}) = (0,1]$, an open subset of $[0,1]$. $x_i \overset{p}{\sim} x_k$, a contradiction. Thus X has only one path component. □

52.11. Theorem. *Continuous image of a path connected space is path connected.*

PROOF. Let X be a path connected space and $f\colon X \to Y$ be an onto continuous function. Then Y is path connected. Let $y_1, y_2 \in Y$. Choose $x_1, x_2 \in X$ such that

$f(x_1) = y_1$ and $f(x_2) = y_2$. Then there is a path γ from x_1 to x_2. So, $f \circ \gamma$ is a path from y_1 to y_2. Thus Y is path connected. □

52.12. Theorem. *A product of topological spaces is path connected if and only if every factor is path connected.*

Proof. Let $\{X_i : i \in I\}$ be a family of topological spaces and $X = \prod_{i\in I} X_i$. Suppose X is path connected. Since each X_i is a continuous image of X, each X_i is path connected.

Conversely, suppose each X_i is path connected. Let $\mathbf{x}, \mathbf{y} \in X$. For every i, by the path connectedness of X_i, there is a path $\gamma_i \colon [0,1] \to X_i$ from $\mathbf{x}_i$ to $\mathbf{y}_i$, $(i \in I)$. So, by the Corollary 22.10, the lifting γ of $\{\gamma_i : i \in I\}$ continuous. Also for every $i \in I$, $\gamma(0)_i = \mathbf{x}_i$ and $\gamma(1)_i = \mathbf{y}_i$. Thus γ is a path from $\mathbf{x}$ to $\mathbf{y}$. □

52.13. Example. If we remove one point from $\mathbb{R}$, then the resulting space is disconnected, and hence, is path disconnected.

Indeed, the set so obtained by removing a point is not an interval, and hence, is disconnected. Consequently, it is path disconnected. We also give a direct proof. For $\mathbb{R} \setminus \{x\}$, choose $a < x < b$. Let $f \colon [0,1] \to \mathbb{R}$ be a path from a to b. Then by the intermediate value theorem, there is $t \in [0,1]$ such that $f(t) = x$. Thus f is not a path in $\mathbb{R} \setminus \{x\}$. That is, $\mathbb{R} \setminus \{x\}$ is path disconnected.

52.14. Example. If we remove a countable set from $\mathbb{R}^2$, then the resulting set is path connected.

Let $A \subset \mathbb{R}^2$ be a countable set. Fix $x, y \in \mathbb{R}^2 \setminus A$. Note that for every $a \in A$, there will is one and one line passing through x and a. Thus there are at most countably many lines passing through x and intersecting A. But as we can there are uncountably many lines passing through x. Thus, there are uncountably many lines passing through x and not intersecting A. In the same way there are uncountably many lines passing through y and not intersecting A. Let L (respectively, M) be the set of all lines passing through x (respectively y) and not intersecting A. If $L \cap M \neq \emptyset$, then there is a line (in fact, a unique line) in $L \cap M$ and that joins x and y. If $L \cap M = \emptyset$, then fix a line ℓ in L. With the grace of Euclid, there is only one line passing through y and parallel to ℓ (and it may not be even in M). So, choose a line in M that is not parallel to ℓ. Then this two lines together join x and y. This completes the proof. Thus $\mathbb{R}^2 \setminus A$ is path connected, and hence, also connected.

Problems

238. Similar to the topologist sine curve, consider the comb space

$$X = (\{\tfrac{1}{n} : n \in \mathbb{N}\} \times [0,1])) \cup (0 \times [0,1]) \cup ([0,1] \times \{0\})$$

and

$$Y = X \setminus (\{0\} \times (0,1)).$$

Show that Y is connected but not path connected.

CHAPTER 10

Uniform Spaces

The uniform spaces were introduced in 1937 by Weil in his book [**Wei37**]. Weil introduced the uniform spaces in the language of entourages or what we have called surroundings. However there is another approach of uniform spaces through uniform coverings. It was introduced by Tukey in 1940 in his book [**Tuk40**].

53. Uniformity

Recall from the Definition 1.8 that for a nonempty set X, the *diagonal*, denoted by Δ or $\Delta(X)$ is the set $\{(x, x) : x \in X\}$. We ask the reader to recall other such notations and definition from the Section 1.

53.1. Definition. Let X be a nonempty set. A *uniformity* or a *diagonal uniformity on* X is a collection $\mathcal{D}$ (or $\mathcal{D}(X)$ – to be specific about X) of subsets of $X \times X$ satisfying the following.

(1) $D \in \mathcal{D} \Rightarrow \Delta \subset D$.
(2) $D_1, D_2 \in \mathcal{D} \Rightarrow D_1 \cap D_2 \in \mathcal{D}$.
(3) $D \in \mathcal{D} \Rightarrow$ there is $E \in \mathcal{D}$ such that $E \circ E \subset D$.
(4) $D \in \mathcal{D} \Rightarrow$ there is $E \in \mathcal{D}$ such that $E^{-1} \subset D$.
(5) $D \in \mathcal{D}, D \subset E \Rightarrow E \in \mathcal{D}$.

We say that $(X, \mathcal{D})$ is a *uniform space* In this case, each $D \in \mathcal{D}$ is called a *surrounding* or *entourage*. We say that $\mathcal{D}$ is *separating* if $\cap_{D \in \mathcal{D}} D = \Delta$. In this case, X is called *separated*.

One easily observes that every diagonal uniformity on a set X is a filter on $X \times X$.

VII. N$^o{}_t$ations: ☞ For $D \subset X \times X$, $D^n = \underbrace{D \circ D \circ \cdots \circ D}_{(n\text{-times})}$.

53.2. Definition. Let X be a set and $\mathcal{E}$ be a collection of subsets of $X \times X$. $\mathcal{E}$ is said to be a *uniformity base* if

$$\mathcal{D} = \{D \subset X \times X : \text{ there is } E \in \mathcal{E} \text{ such that } E \subset D\}$$

is a uniformity on X. A collection $\mathcal{C}$ of subsets of $X \times X$ is called a *uniformity subbase* if the finite intersections of members of $\mathcal{C}$ is a uniformity base.

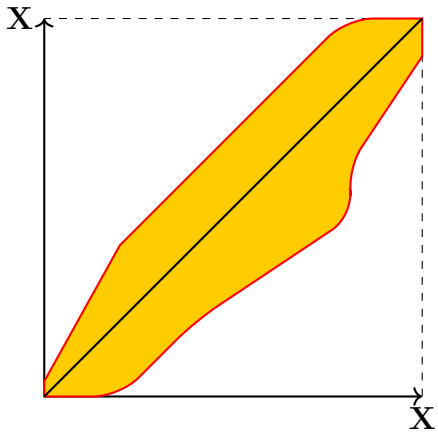


FIGURE 36. A Typical Surrounding

53.3. Theorem. *Let X be a set and $\mathcal{E}$ be a collection of subsets of $X \times X$. Then $\mathcal{E}$ is a uniformity base for a uniformity $\mathcal{D}$ if and only if*

(1) $D \in \mathcal{E} \Rightarrow \Delta \subset D$.
(2) $D_1, D_2 \in \mathcal{E} \Rightarrow$ there is $D_3 \in \mathcal{E}$ such that $D_3 \subset D_1 \cap D_2$.
(3) $D \in \mathcal{E} \Rightarrow$ there is $E \in \mathcal{E}$ such that $E \circ E \subset D$.
(4) $D \in \mathcal{E} \Rightarrow$ there is $E \in \mathcal{E}$ such that $E^{-1} \subset D$.

In this case,

$$\mathcal{D} = \{D \subset X \times X : \text{ there is } E \in \mathcal{E} \text{ such that } E \subset D\} \tag{53.3.1}$$

PROOF. Suppose $\mathcal{E}$ is a uniformity base for a uniformity $\mathcal{D}$ given by (53.3.1). Clearly, $\mathcal{E} \subset \mathcal{D}$. So, every $D \in \mathcal{E}$ contains Δ. Now let $D_1, D_2 \in \mathcal{E}$. So, $D_1, D_2 \in \mathcal{D}$, and hence, $D_1 \cap D_2 \in \mathcal{D}$. So, by the relation in (53.3.1), (2) follows. Let $D \in \mathcal{E}$. So, $D \in \mathcal{D}$. Thus there is $D' \in \mathcal{D}$ such that $D' \circ D' \subset D$. But then there is $E \in \mathcal{E}$ such that $E \subset D'$. Thus $E \circ E \subset D' \circ D' \subset D$. Finally, let $D \in \mathcal{E}$. So, $D \in \mathcal{D}$. Thus there is $D' \in \mathcal{D}$ such that $D'^{-1} \subset D$. But then there is $E \in \mathcal{E}$ such that $E \subset D'$. Thus $E^{-1} \subset D'^{-1} \subset D$.

Conversely, suppose $\mathcal{E}$ satisfies (1)–(4) and define $\mathcal{D}$ by (53.3.1). Since every $D \in \mathcal{E}$ contains Δ, each of its super sets also contains Δ. Let $D_1, D_2 \in \mathcal{D}$. Then there are $E_1, E_2 \in \mathcal{E}$ such that $E_1 \subset D_1$, $E_2 \subset D_2$. But then there is $E \in \mathcal{E}$ such that $E \subset E_1 \cap E_2$. Thus $E \subset D_1 \cap D_1$, and hence, $D_1 \cap D_2 \in \mathcal{D}$. The rest of the two properties follow straight by the similar argument. □

53.4. Example. Let (X, d) be a metric space. For $\varepsilon > 0$, define

$$D_\varepsilon = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}.$$

Then

$$\mathcal{E}_d = \{D_\varepsilon : \varepsilon > 0\} \tag{53.4.1}$$

is a base for a uniformity. D_ε looks as in the Figure 37. The uniformity generated by this uniformity base is called the *metrizable uniformity*. The uniformity generated by the usual metric on $\mathbb{R}^n$ is called the *usual uniformity*.

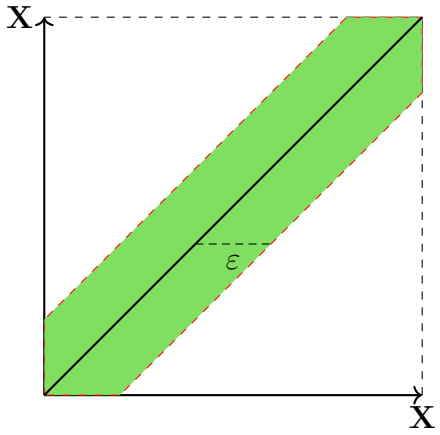


FIGURE 37. D_ε

53.5. Definition. Let X be a nonempty set. A *pseudometric on X* is a function $d\colon X \times X \to \mathbb{R}$ satisfying the following.

(1) $d(x, y) \geq 0$, for all $x, y \in X$ and $d(x, x) = 0$.
(2) $d(x, y) = d(y, x)$, for all $x, y \in X$.
(3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

In this case the set X together with the pseudometric d is called a *pseudometric space*. It is denoted by (X, d) or X.

53.6. Example. Let (X, d) be a pseudometric space. For $\varepsilon > 0$, define

$$D_\varepsilon = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}.$$

Then

$$\mathcal{E}_d = \{D_\varepsilon : \varepsilon > 0\} \tag{53.6.1}$$

is a base for a uniformity. The uniformity generated by this uniformity base is called the *pseudometrizable uniformity*.

53.7. Example. Let X be a nonempty set. The uniformity

$$\mathcal{D} = \{D \subset X \times X : \Delta \subset D\} \tag{53.7.1}$$

is called the *discrete uniformity* on X.

53.8. Example. Let X be a nonempty set. The uniformity

$$\mathcal{D} = \{X \times X\} \tag{53.8.1}$$

is called the *indiscrete uniformity* on X.

53.9. Example. Let X be a linearly ordered set. For $a \in X$, define

$$D_a = \Delta \cup \{(x, y) \in X \times X : x > a, y > a\}.$$

Then $\mathcal{E} = \{D_a : a \in X\}$ is a uniformity base.

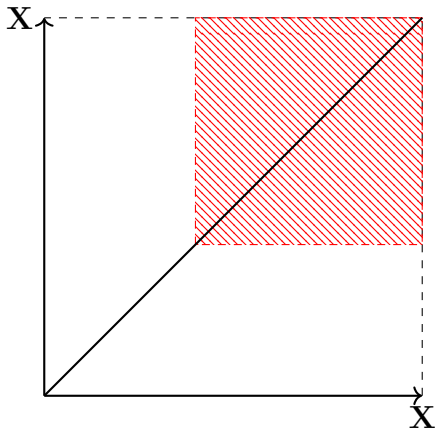


FIGURE 38. D_a

53.10. Proposition. *Let X be a set and $\mathcal{D}$ be a uniformity on X. Then $D \in \mathcal{D} \Leftrightarrow D^{-1} \in \mathcal{D}$.*

PROOF. For $D \in \mathcal{D}$, there is $E \in \mathcal{D}$ such that $E^{-1} \subset D$. So, $E \subset D^{-1}$. Hence $D^{-1} \in \mathcal{D}$. The symmetry of the argument shows that the two way implication holds good. □

53.11. Theorem. *Let X be a nonempty set. Then $\mathcal{D} \subset \mathcal{P}(X)$ is a uniformity on X if and only if*

(1) $D \in \mathcal{D} \Rightarrow \Delta \subset D$.
(2) $D_1, D_2 \in \mathcal{D} \Rightarrow D_1 \cap D_2 \in \mathcal{D}$.
(3) $D \in \mathcal{D} \Rightarrow$ there is $E \in \mathcal{D}$ such that $E \circ E^{-1} \subset D$.
(4) $D \in \mathcal{D}, D \subset E \Rightarrow E \in \mathcal{D}$.

PROOF. Suppose $\mathcal{D}$ is a uniformity on X. Let $D \in \mathcal{D}$. Choose $D_1, D_2 \in \mathcal{D}$ such that $D_1 \circ D_1 \subset D$ and $D_2^{-1} \subset D_1$. Taking $E = D_1 \cap D_2$, we see that $E \in \mathcal{D}$. Also $E \circ E^{-1} \subset D_1 \circ D_2^{-1} \subset D_1 \circ D_1 \subset D$.

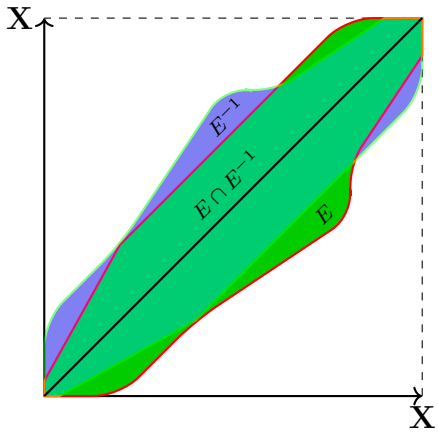


FIGURE 39. $E \cap E^{-1}$

On the other hand assume that all given conditions hold for $\mathcal{D}$. Let $D \in \mathcal{D}$. Find $E \in \mathcal{D}$ such that $E \circ E^{-1} \subset D$. But then $E^{-1} = \Delta \circ E^{-1} \subset E \circ E^{-1} \subset D$. Also, taking $D = E \cap E^{-1}$, it follows that $D \circ D \subset E \circ E^{-1} \subset D$. This completes the proof. □

53.12. Proposition. *Let $\mathcal{D}$ be a uniformity on a set X. Then*

$$\mathcal{E} = \{E \in \mathcal{D} : E = E^{-1}\}$$

is a uniformity base for $\mathcal{D}$.

PROOF. Clearly, $\mathcal{E} \subset \mathcal{D}$. On the other hand, for every $D \in \mathcal{D}$, let $E = D \cap D^{-1}$. Then $E \in \mathcal{E}$ and $E \subset D$. This completes the proof. □

Problems

239. For a metric space (X, d), prove that $\mathcal{E}_d$ defined in (53.4.1) is a uniformity base.

240. Let (X, d) be a metric space. Prove that $D_\varepsilon \circ D_\delta \subset D_{\varepsilon+\delta}$.

241. Let X be a set. Prove that $\mathcal{D}$ defined in (53.7.1) is a uniformity on X.

242. Let X be a set. Prove that $\mathcal{D}$ defined in (53.8.1) is a uniformity on X.

243. Let X be a linearly ordered set. Prove that $\mathcal{E}$ defined in Example 53.9 is a uniformity base on X.

244. Show that $\mathcal{D} = \{D \subset \mathbb{R}^2 : (x, x), (x, -x) \in D \text{ for all } x \in \mathbb{R}\}$ is a uniformity on $\mathbb{R}$.

54. Uniform Topology

We saw that every metric gives a uniformity. Now we proceed to show that every uniformity gives rise to a topology.

54.1. Definition. Let $\mathcal{D}$ be a uniformity on a set X, $x \in X$, $A \subset X$ and $D \in \mathcal{D}$. Then we define

$$D[x] = \{y \in X : (x, y) \in D\} \text{ and}$$
$$D[A] = \cup_{x \in A} D[x].$$

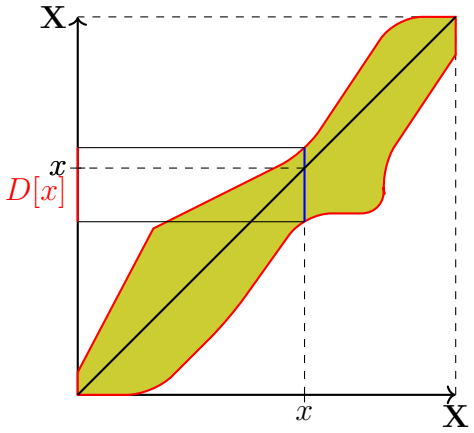


FIGURE 40. $D[x]$

54.2. Theorem. *Let $\mathcal{E}$ be a uniformity base on a set X. Then for every $x \in X$,*

$$\mathscr{B}_x = \{D[x] : D \in \mathcal{D}\}$$

forms a neighbourhood base at x.

PROOF. We prove that (1) to (3) of the Theorem 10.8 are satisfied. Since $\Delta \subset D$ for every $D \in \mathcal{E}$, $x \in D[x]$ for every $D \in \mathcal{E}$. Hence (1) of the Theorem 10.8 follows. Let $D_1, D_2 \in \mathcal{E}$. Choose $D_3 \in \mathcal{E}$ such that $D_3 \subset D_1 \cap D_2$. Then $D_3[x] \subset (D_1 \cap D_2)[x] = D_1[x] \cap D_2[x]$. Since $D_3[x] \in \mathscr{B}_x$, (2) of the Theorem 10.8 follows. Now let $D \in \mathcal{E}$. Chose $E \in \mathcal{E}$ such that $E \circ E \subset D$. Clearly, for any $y \in E[x]$, $(x, y) = (x, y) \circ (x, x) \in E \circ E \subset D$. That is, $y \in D[x]$. Thus $E[y] \subset D[x]$. Consequently, for every $y \in E[x]$, we get $E[y] \in \mathscr{B}_y$ such that $E[y] \subset D[x]$. Hence by the Theorem 10.8, $\mathscr{B}_x$ is a base for some topology on X. □

54.3. Theorem. *Let $\mathcal{D}$ be a uniformity on a set X and $\mathcal{E}$ be a uniformity base for $\mathcal{D}$. Then the local bases given by the $\mathcal{D}$ and $\mathcal{E}$ generate the same topology on X.*

PROOF. It suffices to prove by assuming that $\mathcal{D}$ is a uniformity and $\mathcal{E}$ is its uniformity base. Let $\mathscr{B}_x^{\mathcal{D}}$ and $\mathscr{B}_x^{\mathcal{E}}$ denote the local bases generated by $\mathcal{D}$ and $\mathcal{E}$ respectively. Let $G \in \mathscr{B}_x^{\mathcal{D}}$. So, there is $D \in \mathcal{D}$ such that $G = D[x]$. Since $\mathcal{E}$ is a basis for $\mathcal{D}$, there is $E \in \mathcal{E}$ such that $E \subset D$. But then $E[x] \subset D[x] = G$. Thus $\mathscr{B}_x^{\mathcal{D}}$ generates the coarser topology that the topology generated by $\mathscr{B}_x^{\mathcal{E}}$. The other way is obvious because $\mathscr{B}_x^{\mathcal{E}} \subset \mathscr{B}_x^{\mathcal{D}}$. □

54.4. Definition. Let $\mathcal{D}$ be a uniformity on a set X. The topology on X generated by the local base given by $\mathcal{D}$ is called the *uniform topology on X* and is denoted by $\mathcal{T}_{\mathcal{D}}$.

54.5. Definition. Let $(X, \mathcal{T})$ be a topological space. If there is a uniformity $\mathcal{D}$ on $X \times X$ such that $\mathcal{T} = \mathcal{T}_{\mathcal{D}}$, then $(X, \mathcal{T})$ is called *uniformizable topological space*.

54.6. Example. The topology generated by the usual uniformity on $\mathbb{R}$ is the usual topology. Indeed it follows by noting that for $\varepsilon > 0$ and $x \in \mathbb{R}$,

$$D_\varepsilon[x] = \{y \in \mathbb{R} : (x, y) \in D_\varepsilon\} = \{y \in \mathbb{R} : d(x, y) < \varepsilon\} = (x - \varepsilon, x + \varepsilon).$$

54.7. Example. Let (X, d) be a metric space and $\mathcal{D}_d$ be the metric uniformity. Then the topology generated by $\mathcal{D}_d$ on X is the metric topology given by the metric d. Indeed it follows by noting that for $\varepsilon > 0$ and $x \in X$,

$$D_\varepsilon[x] = \{y \in X : (x, y) \in D\} = \{y \in X : d(x, y) < \varepsilon\} = B_d(x, \varepsilon).$$

54.8. Lemma. *Let $\mathcal{D}$ be a uniformity on a set X and $\mathcal{T}_{\mathcal{D}}$ be the uniform topology on X given by $\mathcal{D}$. If $D \in \mathcal{D}$, then $D^\circ \in \mathcal{D}$.*

Proof. Let $D \in \mathcal{D}$ and let $(x, y) \in E = E^{-1} \in \mathcal{D}$ be such that $E \circ E \circ E \subset D$. Let $(u, v) \in E[x] \times E[y]$. $(x, u), (y, v) \in E$. Since $(x, y) \in E$, we get $(u, v) \in E \circ E \circ E \subset D$. Thus $E[x] \times E[y] \subset D$. But $E[x], E[y]$ are neighbourhoods of x, y respectively in X, and hence, $E[x] \times E[y]$ is a neighbourhood of (x, y) in $X \times X$. That is, D is a neighbourhood of (x, y). Thus $(x, y) \in D^\circ$. This shows that $E \subset D^\circ$. Since $\mathcal{D}$ is a uniformity, we conclude that $D^\circ \in \mathcal{D}$. This completes the proof. □

In fact, we have proved the following stronger result.

54.9. Theorem. *Let $\mathcal{D}$ be a uniformity on a set X and $\mathcal{T}_{\mathcal{D}}$ be the uniform topology on X given by $\mathcal{D}$. Then*

(1) All open elements of $\mathcal{D}$ form a uniformity base for $\mathcal{D}$.
(2) All open symmetric elements of $\mathcal{D}$ form a uniformity base for $\mathcal{D}$.

Proof. The proof follows from the Lemma 54.8. □

Problems

245. Let X be a set. Prove that the indiscrete uniformity on X generates the indiscrete topology on X.

246. Let X be a set. Prove that the discrete uniformity on X generates the discrete topology on X.

247. Let X be a linearly ordered set. For $a \in X$, define $D_a = \{(x, y) \in X \times X : x > a, y > b\}$. Prove that $\{D_a : a \in X\}$ is a uniformity base and it generates the discrete topology.

55. Separation in Uniform Spaces

As we have seen a T_0-space need not be T_1; a T_1-space need not be T_2; a T_3-space need not be T_3 and so on. However, in the uniform spaces, some of the separation axioms coincide with each other.

55.1. Theorem. *Let $\mathcal{D}$ be a uniformity on a set X. The uniform topology $\mathscr{T}_{\mathcal{D}}$ is Hausdorff if and only if $\mathcal{D}$ is separating.*

Proof. Let $(X, \mathscr{T}_{\mathcal{D}})$ be Hausdorff and $(x, y) \in X \times X \setminus \Delta$. Thus $x \neq y$. Hence there are $D, E \in \mathcal{D}$ such that $D[x] \cap E[y] = \emptyset$. This ensures that $D \cap E \in \mathcal{D}$ and $(x, y) \notin D \cap E$. Thus $\mathcal{D}$ is separating.

Conversely, assume that $\mathcal{D}$ is separating. Let $x, y \in X$ be distinct. Then $(x, y) \notin \Delta$. Hence, we get $D \in \mathcal{D}$ such that $(x, y) \notin D$. Choose a symmetric $E \in \mathcal{D}$ such that $E \circ E \subset D$. Then $E[x], E[y]$ are neighbourhoods of x and y respectively. Suppose, if possible, $z \in E[x] \cap E[y]$. But then $(x, z), (y, z) \in E$. By the symmetry of E, $(x, z), (z, y) \in E$, giving $(x, y) \in E \circ E \subset D$, a contradiction. Thus X is Hausdorff. □

55.2. Theorem. *A T_0-uniform space is T_2.*

Proof. Let a uniform space $(X, \mathcal{D})$ be T_0. Let x, y be two distinct points of X. Without loss of generality, let a symmetric $D \in \mathcal{D}$ be such that $y \notin D[x]$. But then $(x, y) \notin D$ and by the symmetry $(y, x) \notin D$. That is, $(x, y), (y, x) \notin D$. Thus $\mathcal{D}$ is separating. Hence by the Theorem 55.1, X is T_2. □

55.3. Lemma. *Let $(X, \mathcal{D})$ be a uniform space, $A \subset X$ and $D \subset \mathcal{D}$. Then $A \subset D[A]^\circ$.*

Proof. For every $x \in A$, $x \in D[x] \subset D[A]$. Since $D[x]$ is a neighbourhood of x, $x \in D[A]^\circ$. Thus $A \subset D[A]^\circ$. □

55.4. Theorem. *A Hausdorff uniform space is regular.*

Proof. Let $(X, \mathcal{D})$ be a uniform space. Let $F \subset X$ be closed and $x \notin F$. So, x is not a limit point of F. Thus there is $D \in \mathcal{D}$ such that $D[x] \cap F =$. Choose a symmetric $E \in \mathcal{D}$ such that $E \circ E \subset D$. It is sufficient to show that $E[x] \cap E[F] = \emptyset$. Let $y \in E[x] \cap E[F]$. Let $z \in F$. So, $z \in F \subset E[F]$. Then $(x, z) = (y, z) \circ (x, y) \in E \circ E \subset D$. Thus $z \in D[x] \cap F$, a contradiction. Since by Lemma 55.3, $F \subset D[F]^\circ$, it follows that $D[x]^\circ$ and $D[F]^\circ$ are disjoint open subsets of X with $x \in D[x]$ and $F \subset D[F]^\circ$. This completes the proof. □

55.5. Corollary. *Let $(X, \mathcal{D})$ be a uniform space and $A, B \subset X$ be disjoint. Suppose A is compact and B is closed. Then there is $D \subset \mathcal{D}$ such that $D[A] \cap D[B] = \emptyset$.*

PROOF. By the Theorem 55.4, for every $x \in A$, there is $E_x \in \mathcal{D}$ such that $x \in E_x[x], B \subset E_x[B]$ and $E_x[x] \cap E_x[B] = \emptyset$. But then $\{E_x[x]^\circ : x \in A\}$ is an open cover for A, and hence, there are $x_1, x_2, x_3, \ldots, x_n \in A$ such that $A \subset \cup_{i=1}^n E_{x_i}$. But then $B \subset \cap_{i=1}^n E_{x_i}[B]$. Taking $D = \cap_{i=1}^n E_{x_i}$, the result follows. □

Now we turn to associating a Hausdorff uniform space with every uniform space as a quotient.

55.6. Definition. Let $(X, \mathcal{D})$ be a uniform space. Define $\sim_{\mathcal{D}}$ on X by $x \sim_{\mathcal{D}} y$ if $(x, y) \in \cap\{D : D \in \mathcal{D}\}$.

55.7. Proposition. *Let $(X, \mathcal{D})$ be a uniform space. Then $\sim_{\mathcal{D}}$ is an equivalence relation on X.*

PROOF. Clearly, $\Delta \subset D$ for every $D \in \mathcal{D}$. Hence, $\sim_{\mathcal{D}}$ is reflexive. Now suppose $x, y \in X$ and $x \sim_{\mathcal{D}} y$. Given a $D \in \mathcal{D}$, choose a symmetric $E \in \mathcal{D}$ such that $E \subset D$. Since $(x, y) \in E$, the symmetry of E gives $(y, x) \in E \subset D$. Thus $(y, x) \in D$ for every $D \in \mathcal{D}$. Let $x, y, z \in X$ such that $x \sim_{\mathcal{D}} y, y \sim_{\mathcal{D}} z$. Let $D \in \mathcal{D}$ choose a symmetric $E \in \mathcal{D}$ such that $E \circ E \subset D$. Clearly, $(y, z), (x, y) \in E$, and hence, $(x, z) = (y, z) \circ (x, y) \in E \circ E \subset D$. Thus $x \sim_{\mathcal{D}} z$. This completes the proof. □

VIII. N$^o{}_t$ations: ☞ Let $(X, \mathcal{D})$ be a uniform space. We fix

$$H = \cap\mathcal{D} = \cap\{D \subset X \times X : D \in \mathcal{D}\}.$$

We shall denote the equivalence class under $\sim_{\mathcal{D}}$ containing x by x' and X_H will denote the quotient space $X/\sim_{\mathcal{D}}$. Also, the quotient map $x \in X \mapsto x' \in X_H$ is denoted by q_H. It is customary to denote the map $(x, y) \in X \times X \mapsto (q_H(x), q_H(y)) \in X_H \times X_H$ by $q_H \times q_H$; see for example [**Jam87**, p. 3 and p 99]. However, we shall use q_H for this map also. It will be clear from the context. For $D \subset X \times X$, we set $D_H = q_H(D)$; as a consequence, $\Delta_H = q_H(\Delta)$.

55.8. Lemma. [**Jam87**, p. 99] *Let $(X, \mathcal{D})$ be a uniform space and $D \in \mathcal{D}$. Then $q_H^{-1}(D') = H \circ D \circ H$.*

PROOF. Let $(x, y) \in q_H^{-1}(D')$. So, $(x', y') \in D'$. Hence there are $(v, w) \in D$ such that $q_H(v, w) = (x', y') = q_H(x, y)$. Clearly, $q_H(x) = q_H(v)$ and $q_H(y) = q_H(w)$. That is, $(w, y), (x, v) \in H$ so that $(x, y) = (w, y) \circ (v, w) \circ (x, v) \in H \circ D \circ H$.

Conversely, suppose $(x, y) = (w, y) \circ (v, w) \circ (x, v) \in H \circ D \circ H$ with $(w, y), (x, v) \in H$ and $(v, w) \in D$. Hence, $q_H(v, w) \in q_H(D) = D'$, $q_H(x) = q_H(v)$

and $q_H(y) = q_H(w)$. Consequently, $q_H(x,y) = (q_H(x), q_H(y)) = (q_H(v), q_H(w)) = q_H(v,w) \in D'$. This proves that $(x,y) \in q_H^{-1}(D')$. □

55.9. Corollary. *If $(X, \mathcal{D})$ is a uniform space and $D \in \mathcal{D}$, then $D \subset D \circ D \circ D$.*

Proof. $D \subset q_H^{-1}(q_H(D)) = q_H^{-1}(D') = H \circ D \circ H \subset D \circ D \circ D$ because $H \subset D$. □

In fact, the more general result can be proved directly.

55.10. Proposition. *For for positive integers $m \leq n$ and $D \in \mathcal{D}$, $D^m \subset D^n$.*

Proof. Note that $D \circ \Delta = D$. So,

$$\begin{aligned} D \circ D \circ \cdots \circ D_{m \text{ times}} &= D \circ D \circ \cdots \circ D_{m \text{ times}} \circ \Delta \circ \Delta \circ \cdots \circ \Delta_{n-m \text{ times}} \\ &\subset D \circ D \circ \cdots \circ D_{n \text{ times}} \end{aligned}$$

□

55.11. Proposition. *Let $(X, \mathcal{D})$ be a uniform space. Then*

$$\mathcal{C} = \{H \circ D \circ H : D \in \mathcal{D}\}$$

forms a base for $\mathcal{D}$.

Proof. For every $D \in \mathcal{D}$, $\Delta \subset H \subset D$, and hence, $\Delta \subset H \circ D \circ H$. Let $D, E \in \mathcal{D}$ and $(x,y) \in H \circ (D \cap E) \circ H$. So, $(x,y) = (w,y) \circ (v,w) \circ (x,v)$ with $(x,v), (w,y) \in H$ and $(v,w) \in D \cap E$. So, $(v,w) \in D$ and $(v,w) \in E$. Hence $(x,y) \in H \circ D \circ H$ and $(x,y) \in H \circ E \circ H$. Thus $H \circ (D \cap E) \circ H \subset (H \circ D \circ H) \cap (H \circ E \circ H)$.

Now let $D \in \mathcal{D}$. So, there is $E \in \mathcal{D}$ such that $E \circ E \subset D$. Then using $H \circ H = H$, $H \circ E \circ H \subset E$ and $E \circ E \subset D$, we get

$$\begin{aligned} (H \circ E \circ H) \circ (H \circ E \circ H) &= H \circ E \circ H \circ E \circ H \\ &= H \circ E \circ H \circ E \circ H \circ H \\ &\subset H \circ E \circ E \circ H \\ &\subset H \circ D \circ H. \end{aligned}$$

Finally, for $D \in \mathcal{D}$, $(H \circ D \circ H)^{-1} = H^{-1} \circ D^{-1} \circ H^{-1} = H \circ D^{-1} \circ H \in \mathcal{C}$. Thus by the Theorem 53.3, $\mathcal{C}$ is a uniformity base for $\mathcal{D}$. □

55.12. Proposition. *Let $(X, \mathcal{D})$ be a uniform space. On X_H define*

$$\mathcal{B}_H = \{q_H(D) : D \in \mathcal{D}\}. \tag{55.12.1}$$

Then

(1) $\mathcal{B}_H$ is a uniformity base for a separating uniformity on X_H.
(2) $q\colon X \to X_H$ is uniformly continuous and the uniform topology on X_H is the quotient topology given by q.

PROOF. ☞(1) For every $x' \in X_H$ and $q_H(D) \in \mathcal{B}_H$, $(x,x) \in D$, and hence, $(x',x') \in D_H$. Thus $\Delta_H \subset D_H$ for every $D_H \in \mathcal{B}_H$.

Consider $E_H, F_H \in \mathcal{D}_H$, where $E, F \in \mathcal{D}$. Clearly,

$$(E \cap F)_H = q_H(E \cap F) \subset q_H(E) \cap q_H(F) = E_H \cap F_H.$$

Let $D_H \in \mathcal{B}_H$ with $D \in \mathcal{D}$. Choose $E_1, E_2, E \in \mathcal{D}$ such that $E_1^2 \subset D, E_2^2 \subset E_1$ and $E^2 \subset E_2$. So, $E^6 \subset E^8 \subset E_2^4 \subset E_1^2 \subset D$. Now let $(x',y') \in E_H \circ E_H$, where $q_H(x) = x'$ and $q_H(y) = y'$. Choose $z' = q_H(z) \in X'$ such that $(x',z'), (z',y') \in E_H$. Consequently, $(x,z), (z,y) \in H \circ E \circ H$ so that $(x,y) \in H \circ E \circ H \circ H \circ E \circ H \subset E^6 \subset D$. Thus $(x',y') \in D_H$. That is, $E_H \circ E_H \subset D_H$.

Finally, let $D_H \in \mathcal{B}_H$ with $D \in \mathcal{D}$. But then there is $E \in \mathcal{D}$ such that $E^{-1} \subset D$. Hence $E_H^{-1} \subset D_H^{-1}$.

Summing up all this and using the Theorem 53.3, we conclude that $\mathcal{B}_H$ is a uniformity base for a uniformity on X_H. Now let $(x',y') \in X_H$ such that $q_H(x) = x' \neq y' = q_H(y)$. That is, $x \not\sim_{\mathcal{D}} y$. Hence $(x,y) \notin H$. Let $D \in \mathcal{D}$ be such that $(x,y) \notin D$. But then $(x',y') \notin D_H$. Hence the uniformity generated by $\mathcal{B}_H$ is separating.

☞ (2) Let D_H be in $\mathcal{D}_H$, then $q_H(D) \subset D_H$ and hence q_H is uniformly continuous. Also, let $U \subset X'$ be such that $q_H^{-1}(U)$ is open in X and $q_H(x) \in U$. So, there is $D \in \mathcal{D}$ such that $D[x] \subset q_H^{-1}(U)$. Let $y' \in D_H[x']$. So, $(x',y') \in D_H$. But then there is $z \in X$ such that $q_H(z) - y'$ and $(x,z) \in D$. Thus $z \in D[x]$ and so $y' \in U$. Thus U is open the uniform topology. That is, $\mathcal{B}_H$ generates the quotient topology given by q_H.

□

55.13. Definition. Let $\mathcal{D}$ be a uniformity on a set X. The uniformity $\mathcal{D}_H$ on X_H generated by the uniformity base $\mathcal{B}_H$ defined in the Proposition P:032 is called the *Hausdorff quotient of* $\mathcal{D}$.

56. Convergence in a Uniform Space

In this section, we extend the concept of Cauchy sequences in metric spaces to the Cauchy nets in uniform spaces.

56.1. Definition. Let $(X, \mathcal{D})$ be a uniform space and $\{x_\lambda\}$ be a net in X. We say that $\{x_\lambda\}$ is *Cauchy* if for every $D \in \mathcal{D}$, there is λ_0 such that $(x_\lambda, x_\mu) \in D$ whenever $\lambda, \mu \geq \lambda_0$; that is, in other words, $T_{\lambda_0}(\{x_\lambda\}) \times T_{\lambda_0}(\{x_\lambda\}) \subset D$.

Recalling Note

In case of a convergent net $x_\lambda \to x$, never ever write $\lim_{n\to\infty} x_\lambda = x$ or $x_\lambda \to x$ as $\lambda \to \infty$. This is meaningless. Think what happens when $\Lambda = \{1,2,3,4\}$ with the usual order.

56.2. Proposition. *Every convergent net is Cauchy.*

PROOF. Let $\{x_\lambda\}$ be a net in a uniform space $(X, \mathcal{D})$ converging to x. Let $D \in \mathcal{D}$. Choose a symmetric $E \in \mathcal{D}$ such that $E \circ E \subset D$. Since $x_\lambda \to x$, there λ_0 such that $(x_\lambda, x) \in E$ for every $\lambda \geq \lambda_0$. So, $(x_\lambda, x_\mu) = (x, x_\mu) \circ (x_\lambda, x) \in E \circ E \subset D$ for $\lambda, \mu \geq \lambda_0$,. Thus $\{x_\lambda\}$ is Cauchy. □

Clearly, the converse is not true as that is the case for sequences too and every sequence is a net. Similar to complete metric spaces, we have the following definition.

56.3. Definition. A uniform space $(X, \mathcal{D})$ is said to be *complete* if every Cauchy net in X is also convergent.

The concept of Cauchy net is easily generalized by observing the Cauchy sequence. However, the concept of Cauchy filter is not so apparent.

56.4. Definition. Let $(X, \mathcal{D})$ be a uniform space. A filter $\mathscr{F}$ on X is said to be *Cauchy* if for every $D \in \mathcal{D}$, there is $F \in \mathscr{F}$ such that $F \times F \subset D$.

The first step is to prove that every convergent filter in a uniform space is Cauchy.

56.5. Proposition. *Let $(X, \mathcal{D})$ be a uniform space and $\mathscr{F}$ be a convergent filter. Then $\mathscr{F}$ is Cauchy.*

PROOF. Let $x \in X$ be such that $\mathscr{F} \to x$. Let $D \in \mathcal{D}$. Choose a symmetric $E \in \mathcal{D}$ such that $E \circ E \subset D$. Since $E[x] \in \mathscr{U}_x$ and $\mathscr{F} \to x$, $E[x] \in \mathscr{F}$. Let $y, z \in E[x]$. So, $(x, y), (x, z) \in E$. By the symmetry of E, $(y, x) \in E$. Thus $(y, z) = (x, z) \circ (y, x) \in E \circ E \subset D$. Thus $E[x] \times E[x] \subset D$. So, $\mathscr{F}$ is Cauchy. □

56.6. Example. Let $X = (0, 1)$ with the usual metric uniformity and $\mathscr{F} = \{F \subset X : (0, \varepsilon) \subset E \text{ for some } \varepsilon > 0\}$. Then $\mathscr{F}$ is a Cauchy filter that does not converge.

56.7. Theorem. *Let $(X.\mathcal{D})$ be a uniform space. A net $\{x_\lambda\}_{\lambda \in \Lambda}$ in X is Cauchy if and only if the filter $\mathscr{F}(\{x_\lambda\})$ generated by $\{x_\lambda\}$ is Cauchy.*

PROOF. Suppose $\{x_\lambda\}$ is Cauchy. Let $D \in \mathcal{D}$. Then there is $\mu \in \Lambda$ such that $T_\mu(\{x_\lambda\}) \times T_\mu(\{x_\lambda\}) \subset D$. Since $T_\mu(\{x_\lambda\}) \in \mathscr{F}$, we conclude that $\mathscr{F}$ is Cauchy.

Conversely, suppose $\mathscr{F}(\{x_\lambda\})$ is Cauchy. Let $D \in \mathcal{D}$. So, there is $F \in \mathscr{F}(\{x_\lambda\})$ such that $F \times F \subset D$. But since $\mathscr{F}(\{x_\lambda\})$ is generated by $\{x_\lambda\}$, there is $T_\mu(\{x_\lambda\}) \subset F$. Thus $T_\mu(\{x_\lambda\}) \times T_\mu(\{x_\lambda\}) \subset D$. This completes the proof. □

56.8. Theorem. *Let $(X, \mathcal{D})$ be a uniform space. A filter $\mathscr{F}$ on X is Cauchy if and only the net based on $\mathscr{F}$ is Cauchy.*

PROOF. Suppose $\mathscr{F}$ is Cauchy. Let $D \in \mathcal{D}$. Then there is $E \in \mathscr{F}$ such that $E \times E \subset D$. Choose $b \in E$. Then $T_{(b,E)}(\{x_{(a,F)}\}) \times T_{(b,E)}(\{x_{(a,F)}\}) \subset E \times E \subset D$.

Conversely, suppose the net $\{x_{(a,F)}\}$ is Cauchy. S0, there is $(b, E) \in \Lambda_{\mathscr{F}}$ such that $T_{(b,E)}(\{x_{(a,F)}\}) \times T_{(b,E)}(\{x_{(a,F)}\}) \subset D$. But then $E \times E \subset D$. This completes the proof. □

All the above discussion combined with the Theorems 36.3 and 36.9 gives the following.

56.9. Corollary. *A uniform space $(X, \mathcal{D})$ is complete if and only if every Cauchy filter in X is convergent.*

57. Uniformly Continuous Functions

In this section, we define a *uniformly continuous function* and show that in metric spaces the concept coincides with the already defined uniformly continuous functions between metric spaces.

57.1. Definition. Let $(X, \mathcal{D}), (Y, \mathcal{E})$ be uniform spaces. A function $f : X \to Y$ is said to be *uniformly continuous* if for every $E \in \mathcal{E}$ there is $D \in \mathcal{D}$ such that

$$(x, y) \in D \Rightarrow (f(x), f(y)) \in E.$$

If f is bijective, and f and f^{-1} are uniformly continuous, then f is called *uniform isomorphism* or *uniform equivalence*. In this case, $(X, \mathcal{D})$ and $(Y, \mathcal{E})$ are called *uniformly isomorphic* or *uniformly equivalent*.

We begin by easy observations and results.

57.2. Proposition. *Every uniformly continuous function is continuous.*

PROOF. Let $(X, \mathcal{D}), (Y, \mathcal{E})$ be uniform spaces and $f : X \to Y$ be a uniformly continuous function. Let $x \in X$ and $y = f(x)$. Let $V \subset Y$ be a neighbourhood of y. So, there is $E \in \mathcal{E}$ such that $E[y] \subset V$. By the uniform continuity of f, there is $D \in \mathcal{D}$ such that $(a, b) \in D \Rightarrow (f(a), f(b)) \in E$. Let $a \in D[x]$. So, $(x, a) \in D$. Hence, $(f(x), f(a)) \in E$. Thus $f(a) \in E[y] \subset V$. This completes the proof. □

The above Proposition shows that uniform continuity is a global continuity. In fact, the following proposition shows that the concept of uniform continuity defined in the Definition 57.1 is the same as that defined in the Definition 28.8.

57.3. Proposition. *Let $(X,d),(Y,\rho)$ be metric spaces and $f\colon X\to Y$ be a function. Let $\mathcal{D}_d$ and $\mathcal{D}_\rho$ be the uniformities on X and Y generated by the metrics d and ρ respectively. Then f is uniformly continuous with respect to d and ρ if and only if f is uniformly continuous with respect to $\mathcal{D}_d$ and $\mathcal{D}_\rho$.*

Proof. The proof follows by noting that

$$d(x,y)<\delta \Leftrightarrow (x,y)\in D_{d,\delta}=\{(x,y)\in X\times X : d(x,y)<\delta\}; \tag{57.3.1}$$

and

$$\rho(f(x),f(y))<\varepsilon \Leftrightarrow (f(x),f(y))\in D_{\rho,\varepsilon}=\{(a,b)\in Y\times Y : d(a,b)<\varepsilon\}. \tag{57.3.2}$$

□

58. Supplementary Exercises

Problems

248. Let X be a set and $A\subset X$. Define

$$\mathscr{T}_{A,f}=\{G\subset X : X\setminus G \text{ is finite or} A\cap G=\emptyset\}. \tag{58.0.1}$$

Show that

(1) $\mathscr{T}_{A,f}$ is a topology on X.
(2) Every subset of $X\setminus A$ is $\mathscr{T}_{A,f}$-open.
(3) A is $\mathscr{T}_{A,f}$-closed.
(4) A is $\mathscr{T}_{A,f}$-open if and only if $X\setminus A$ is finite.
(5) If $A=\emptyset$, then show that $\mathscr{T}_{A,f}=\mathscr{D}$.
(6) If X is finite, then show that $\mathscr{T}_{A,f}=\mathscr{D}$.
(7) $\mathscr{T}_{A,f}\leq\mathscr{T}_f$.

249. Let α be an infinite cardinal. Show that there are spaces in which the topology and the continuity are characterized using α-nets.

250. Let α be an infinite cardinal. Given an example of a topological space in which α-nets are not sufficient to characterize the topology and the continuity.

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Mathematicians

Here is the list of the mathematicians whose names appeared in our text.

A

Alexandroff: Pavel Sergeyevich Alexandrov (transliterated in Roman alphabet as Alexandroff) 1896–1982; Russia 2, 210

B

Baire: René-Louis Baire 1874–1932; France 2
Betti: Enrico Betti Glaoui 1823–1892; Italy 1
Bolzano: Bernardus Placidus Johann Nepomuk Bolzano *aka* Bernard Bolzano 1781–1848; Bohemia (now Czech Republic) 1
Borel: Félix Edouard Justin Émile Borel 1871–1956; France 2

C

Cantor: Georg Ferdinand Ludwig Philipp Cantor 1845–1918; Germany 1
Cathcart: George Lambert Cathcart; Germany 2
Cauchy: Augustin Louis Cauchy 1789–1857; France 114, 146, 232
Čech: Eduard Čech 1893–1960; Czechoslovakia, now Czech Republic 211

D

De Morgan: Augustus De Morgan 1806–1871; Britain (born in Madurai, India) 21

E

Euclid: Euclid of Alexandria 325 BC–265BC; Greece 221
Euler: Leonhard Euler 1707–1783; Switzerland 1

F

Fréchet: Maurice René Fréchet 1878–1973; France 1, 2, 150

H

Harnack: Karl Gustav Axel von Harnack 1851–1888, Germany 2
Hausdorff: Felix Hausdorff 1868–1942; Germany 1, 2, 156
Heegaard: Poul Heegaard 1871–1948; Denmark 1
Hilbert: David Hilbert 1862–1943; Germany 1, 2
Hölder: Ludwig Otto Hölder 1859–1937; Germany 105

J

Jordan: Marie Ennemond Camille Jordan 1838–1922; France 1

K

Kolmogorov: Andrey Nikolaevich Kolmogorov 1903–1987; Russia 156

L

L'Huilier: Simon Antoine Jean L'Huilier 1750–1840; Switzerland 1
Lipschitz: Rudolf Otto Sigismund Lipschitz 1832–1903;East Prussia now Russia 130
Listing: Johann Benedict Listing 1802–1882; Germany 2

M

Möbius: August Ferdinand Möbius 1790-1868; Germany 1
Moore: Robert Lee Moore 1882–1974; America 44

N

Ninkowski: Hermann Minkowski 1864–1909; Russian Empire later Lithuania 105

P

Poincaré: Jules Henri Poincaré 1854–1912; France 1

R

Riemann: Georg Friedrich Bernhard Riemann 1826–1866; Germany 1, 141

S

Sierspińsky: Wacław Franciszek Sierspińsky 1868–1942; Germany 37
Sorgenfrey: Robert Henry Sorgenfrey 1915–1995; America 2
Stone: Marshall Harvey Stone 1903–1989; America 211

T

Tait: Peter Guthrie Tait 1931–1901; Scotland 2
Tietze: Heinrich Franz Friedrich Tietze 1880–1964; Germany 156
Tychonoff: Andrey Nikolaevich Tikhonov (transliterated in Roman alphabet as Tychonoff)1906–1993; Soviet and later Russia 2
Tychonoff: Andrey Nikolaevich Tikhonov 1906–1993; Soviet and later Russia 156

U

Urysohn: Pavel Samuilovich Urysohn 1898–1924; Ukraine 2

W

Weierstrass: Karl Theodor Wilhelm Weierstrass 1815–1897; Germany 1
Weil: André Abraham Weil 1906–1998; France 222

Z

Zermelo: Ernst Friedrich Ferdinand Zermelo 1871–1953; Germany 2, 30

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